Conformal Field Theory and the Long-Range Ising Model

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Conformal field theory (CFT) is a powerful framework for elucidating universal properties and describing critical phenomena in various systems, ranging from statistical mechanics to quantum field theory. One hint of conformal symmetry is scale invariance, which is typically observed at critical fixed points of renormalization group flows for which correlation lengths are infinite and there is no length scale. While conformal symmetry always entails scale invariance, the converse has only been shown to be true under certain assumptions, making proofs of conformal invariance non-trivial for some theories. In this work, we provide a brief overview of CFT in $d \geq 3$ dimensions, and go over essential ideas from quantum field theory (QFT) and the renormalization group. These elements are then applied to the study of conformal invariance in the long-range Ising model (LRI). This model can be viewed as a non-local version of the usual Ising model – referred to as the short-range Ising model (SRI) here – where nearest-neighbor interactions are replaced by interactions which go like $1/r^{d+\sigma}$. Just like the SRI, the LRI displays a second-order phase transition at a critical temperature. However, the critical theory depends on a parameter σ. For $\sigma > \sigma^* = 2 - \eta_{\mathrm{SRI}}$, using the usual η critical exponent from the Ising model, the critical theory reduces to the SRI, and for $\sigma < d/2$, the critical theory is Gaussian. The intermediate case, $d/2 < \sigma < \sigma^*$, is a non-trivial and non-Gaussian theory. Using the ε-expansion, we study the theory close to $\sigma = d/2$, in the so-called LRI fixed point, which can be described using a generalized free field perturbed by a marginally relevant ϕ^4 interaction. We then calculate certain correlation functions to test conformality, and justify conformal invariance to all orders in perturbation theory. This study is essential in paving the way for the application of conformal bootstrap methods, which yield non-perturbative approaches to constraining CFT data and effectively solving theories.

Contents

1 Introduction

One of the biggest unresolved problems in theoretical physics is the quantization of gravity. Indeed, fitting gravity into a quantum field-theoretic framework has proven challenging from a conceptual standpoint, not to mention the sophistication of the formalism and the lack of experimental evidence. For some, the main culprit behind the difficulty of the problem is quantum mechanics and QFT [1]. Indeed, even the most basic situations in QFT involve complex functional integrals, canceling infinities through a process called renormalization, and interpreting asymptotic series whose convergence is far from guaranteed. Moreover, most computations in QFT are carried out via perturbation theory, assuming weakly-coupled interactions. However, some phenomena are fundamentally non-perturbative and one must resort to other techniques to study them (typically regarding the strong force and confinement). Therefore, there is some consensus that, in order to develop a theory of quantum gravity, further understanding – particularly non-perturbative – of QFT will be necessary. Incidentally, putting aside gravity, furthering such an understanding has intrinsic worth, since QFT is often considered the most fundamental theory of nature.

A quantum field can be constructed using unitary representations of the Poincaré group (which have to be infinite-dimensional since it is a non-compact group). However, one might want to add additional symmetries to further constrain the theory. A group of symmetries which has proven to be particularly useful is the conformal group, whose study yields conformal field theory (CFT). A CFT is a theory which is invariant under the conformal group i.e. the group containing any transformation which "preserves angles". This property is summarized by a constraint on how the metric changes under such a transformation $x \mapsto \tilde{x}$:

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) = \Lambda(x) g_{\mu\nu}(x) \tag{1.1}$$

where Λ is a function called the *scale factor*. Notice that this is consistent with the geometric picture of angles being preserved. Indeed for two vectors $\mathbf{v}, \mathbf{w}$, the following ratio is clearly preserved under a conformal transformation (simply put, the Λ factors cancel out):

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{g_{\mu\nu} v^\mu w_\nu}{\sqrt{g_{\mu\nu} v^\mu v^\nu}\sqrt{g_{\mu\nu} w^\mu w^\nu}} \tag{1.2}$$

In particular, the case $\Lambda = 1$ corresponds to the Poincaré group, which is a subgroup of the conformal group in flat spacetime (we won't be considering curvature here). Hence, representations of the conformal group will yield a subclass of QFTs.

It turns out that many theories exhibit conformal symmetry. Examples of these include sourceless quantum electrodynamics (QED), massless Dirac fields or even Yang-Mills theory in $d = 4$ dimensions (a gauge theory based on $SU(N)$ useful for understanding the standard model). Moreover, one knows from statistical mechanics that, close to a critical fixed point of a renormalization group (RG) flow, the correlation length of given degrees of freedom is infinite. This implies the absence of a length scale, and hence scale invariance (invariance under $x \mapsto \lambda x$). Scale invariance has been shown to imply conformal symmetry for $d = 2$, and this is often the case for $d > 2$. Thus, a critical fixed point of an RG flow can typically be a CFT. Since the study of fixed points is at the heart of the renormalization group and can yield critical exponents for entire universality classes, CFT can prove very useful to the study of a given QFT even when it isn't conformally invariant, since once can study it close to a critical point. Furthermore, through the so-called *conformal bootstrap* program, CFT allows one to obtain numerical constraints on the fundamental "data" of a theory, thus allowing one to get closer to solving it non-perturbatively, which is one of the main goals in current QFT research.

In this work, we start by reviewing the basic elements of CFT and QFT needed to understanding recent work conducted on the long-range Ising model (LRI for short). The LRI is a non-local extension of the regular Ising model, in that it includes infinite-range interactions going like an inverse power of distance between spins. It turns out that at two special "crossover" points, the LRI can be shown to be conformally invariant. As an instructive application of the first sections on CFT and QFT, we review some of the arguments supporting conformal invariance in one of these points, with a particular emphasis on a paper by Paulos et al. [2].

2 Introduction to Conformal Field Theory

2.1 The conformal group in $d \geq 3$

In this section, we derive all of the infinitesimal conformal transformations for $d \geq 3$ in a flat spacetime of signature (p, q), following [3, 4]. Consider an infinitesimal transformation defined by $\tilde{x}^\mu = x^\mu + \varepsilon^\mu$. Plugging this into equation (1.1) yields:

$$\left(\delta_\mu^\alpha - \partial_\mu \varepsilon^\alpha\right)\left(\delta_\nu^\beta - \partial_\nu \varepsilon^\beta\right)\eta_{\alpha\beta} = \Lambda \eta_{\mu\nu} \tag{2.1}$$

To first order in ε:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \kappa \eta_{\mu\nu} \tag{2.2}$$

where $\kappa = 1 - \Lambda$. Taking the trace of the above equation yields

$$\kappa = \frac{2}{d} \partial \cdot \varepsilon \tag{2.3}$$

Thus κ can be expressed using the divergence of the infinitesimal transformation, which we denote $\partial \cdot \varepsilon := \partial_\rho \varepsilon^\rho$. Therefore, the constraint on ε can be written as

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}\left(\partial \cdot \varepsilon\right)\eta_{\mu\nu} \tag{2.4}$$

Acting with ∂_ρ on the above equation and taking circular permutations of (μ, ν, ρ) leads to

$$\partial_\rho \partial_\nu \varepsilon_\mu = \frac{1}{d}\left(\eta_{\mu\nu}\partial_\rho + \eta_{\mu\rho}\partial_\nu - \eta_{\nu\rho}\partial_\mu\right)\left(\partial \cdot \varepsilon\right) \tag{2.5}$$

Next, we apply ∂^μ:

$$(d-2)\partial_\rho \partial_\nu \left(\partial \cdot \varepsilon\right) = -\eta_{\nu\rho} \Box \left(\partial \cdot \varepsilon\right) \tag{2.6}$$

Taking the trace of this equation gives $(d-1)\Box\left(\partial \cdot \varepsilon\right) = 0$, and plugging this back into the above equation yields the following condition on infinitesimal conformal transformations:

$$(d-1)(d-2)\partial_\rho \partial_\nu \left(\partial \cdot \varepsilon\right) = 0 \tag{2.7}$$

Here, one needs to pay attention to the value of d. For $d = 1$, this doesn't yield any additional information. In fact, our definition of a conformal transformation also no longer yields additional information. It turns out that one-dimensional CFT can be defined via conformal quantum mechanics, which we won't discuss here. For $d = 2$, this doesn't yield a useful constraint either. However, using results from complex analysis, one can deduce that any analytical transformation (holomorphic function) will be conformal in two dimensions. This gives rise to an infinite family of generators, and leads to Virasoro algebras, which we again do not broach upon in this work for the sake of brevity (although $d = 2$ is a case where CFT is in fact extremely useful, scale invariance implying conformal invariance among other things). Hence, focusing on $d \geq 3$ gives the following condition:

$$\partial_\rho \partial_\nu \left(\partial \cdot \varepsilon\right) = 0 \tag{2.8}$$

One deduces from the above relation that ε can be at most quadratic in x, hence the following ansatz:

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{2.9}$$

where each coefficient is of order $o(1)$, and in particular $c_{\mu\nu\rho}$ is symmetric in (ν, ρ). The term a_μ is clearly related to translations. Applying the constraint (2.4) to the second term implies that $b_{\mu\nu}$ has a symmetric part proportional to the metric. Hence, denoting its antisymmetric part $m_{\mu\nu}$, we get $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$. The symmetric part corresponds to a dilatation, while the antisymmetric part corresponds to a Lorentz transformation. Finally, the constraint applied to the quadratic term gives rise to the so-called "special conformal transformation" (SCT):

$$\tilde{x}^\mu = x^\mu + 2(x \cdot b)x^\mu - x^2 b^\mu \tag{2.10}$$

where $b_\mu = c^\nu_{\nu\mu}/d$. We've classified all of the infinitesimal transformations, which give rise to finite transformations through integration. Let's derive the finite SCT as this is useful for deriving two results in this work (see section 2.4 and appendix B.2). Consider a scale transformation with $b = tb_0$. The above equation holds when t is small, and this yields a differential equation:

$$\dot{x} = 2(x \cdot b_0)x - x^2 b_0 \tag{2.11}$$

where x and b_0 are vector quantities. Defining $y = x/x^2$, one can easily show that in fact $\dot{y} = -b_0$, effectively solving the equation. Hence, for a finite SCT parameter b, one has

$$\tilde{x}^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2(b \cdot x) + b^2 x^2} \tag{2.12}$$

Incidentally, this can be shown to be the composition of an inversion of x, followed by a translation by b, followed by another inversion, where an inversion is here defined as $x^\mu \mapsto x^\mu/|x|^2$.

2.2 The conformal algebra

It is straightforward to determine the generators of translations, Lorentz transformations, dilatations and SCTs respectively:

$$\begin{cases} P_\mu = -i\partial_\mu \\ L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \\ D = -ix^\mu \partial_\mu \\ K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) \end{cases} \tag{2.13}$$

As an example, let's detail the derivation of K_μ. By definition (see equation (2.10)), The Killing vector ξ_μ associated with a SCT is given by:

$$b \cdot \xi_\mu = 2x_\mu x^\nu b_\nu - x^2 b_\mu \tag{2.14}$$

Using ideas from differential geometry, one can take a natural basis $\{\partial_\mu\}$ such that $b = b^\mu \partial_\mu$.

$$\xi_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \tag{2.15}$$

As a convention, infinitesimal transformations generated by an operator T are written as $1 - i\varepsilon T$ when $\varepsilon \to 0$. Hence the following generator of SCTs:

$$K_\mu = -i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) \tag{2.16}$$

Furthermore, these generators form a Lie algebra, which is called the conformal algebra. It is characterized by the following commutation relations, the missing commutators being zero:

$$
\begin{cases}
[D, P_\mu] = iP_\mu \\
[D, K_\mu] = -iK_\mu \\
[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \\
[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})
\end{cases}
\tag{2.17}
$$

In particular, it is worth noting that except for Lorentz transformations, all transformations of the same nature commute with one another. Moreover, the first two commutation relations are conceptually important. Indeed, they show that P_μ and K_μ act like creation and annihilation operators on eigenspaces of D, and this will be useful in the section 2.3. Again, let's derive one of these relations as an example.

$$
\begin{aligned}
[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) &= [-i\left(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu\right), -i\partial_\nu] \\
&= [\partial_\nu, \left(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu\right)] \\
&= 2[\partial_\nu, x_\mu x^\rho \partial_\rho] - [\partial_\nu, x^2 \partial_\mu] \\
&= 2[\partial_\nu, x_\mu x^\rho]\partial_\rho - [\partial_\nu, x^2]\partial_\mu \\
&= 2\eta_{\mu\alpha}\left(\delta^\alpha_\nu x^\rho + x^\alpha \delta^\rho_\nu\right)\partial_\rho - 2x_\nu \partial_\mu \\
&= 2\eta_{\mu\nu} x^\rho \partial_\rho + 2(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
&= 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right)
\end{aligned}
\tag{2.18}
$$

In $d \geq 3$ dimensions, there is one dilatation, d translations, d SCTs and $d(d-1)/2$ rotations. These sum up to $(d+1)(d+2)/2$ generators, which incidentally coincides with the number of generators of $\mathfrak{so}(d+2)$. In fact, one can show that the conformal algebra coincides with $\mathfrak{so}(p+1, q+1)$ for $d = p+q$. This can be done by defining the following family of generators:

$$
\begin{cases}
J_{\mu,\nu} = L_{\mu\nu} \\
J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right) \\
J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right) \\
J_{-1,0} = D
\end{cases}
\tag{2.19}
$$

Using the previous commutation relations leads to the following commutation relations, which summarize all there is to know about the conformal algebra:

$$
[J_{mn}, J_{pq}] = i\left(\eta_{mq} J_{np} + \eta_{np} J_{mq} - \eta_{mp} J_{nq} - \eta_{nq} J_{mp}\right)
\tag{2.20}
$$

where the metric used is of signature $(p+1, q+1)$. Therefore, the conformal algebra is $\mathfrak{so}(p+1, q+1)$, and we shall sometimes use the shorthand $SO(p+1, q+1)$ for the conformal group (although strictly speaking having the same Lie algebra as another group doesn't entail equality with that group, unless it is simply-connected).

2.3 Representations of the conformal group

Fields, which become operators in canonical quantization, transform in irreducible unitary representations of the Poincaré group, which are infinite-dimensional since it's non-compact. For a scalar quantum field ϕ (we won't be dealing with fields endowed with spin in the LRI) and $\Lambda \in SO(p, q)$, we have the following action:

$$
U(\Lambda)^\dagger \hat{\phi}(x) U(\Lambda) = \hat{\phi}(\Lambda^{-1} x)
\tag{2.21}
$$

Wherever QFT is involved in this work, we'll be using the path integral formalism, which is equivalent to canonical quantization but enables one to use *classical* field configurations, the quantum fluctuations being encoded into a Boltzmann-like probability distribution. Hence, we'll be studying classical fields from here on out.

Since we're working with irreducible representations of the Lorentz group and we've previously seen that $[D, L_{\mu\nu}] = 0$, Schur's lemma implies the existence of a scalar Δ, called the *scaling dimension* of the field, such that $D := -i\Delta\,\mathrm{Id}$. The value of $[D, K_\mu]$ then implies that $K_\mu \phi = 0$ i.e. the field is annihilated by the generator of SCTs. Fields, or operators which satisfy this property are called *primary operators*. If an operator is obtained from another by acting with generators of the conformal group, it is called a *descendant* *operator.*

Next, we derive the action of this algebra on $\phi(x)$, given its behavior at $x=0$ i.e. $L_{\mu\nu}, \phi(0) = S_{\mu\nu}\phi(0)$ (this conceptually amounts to first considering a "little group" preserving the origin [5]). Using the translation operator, the Baker-Hausdorff lemma, and the conformal commutation relations one can easily check that (see [6–8] for the equivalent statement in the operator formalism):

$$\begin{aligned} L_{\mu\nu}\phi(x) &= L_{\mu\nu}e^{-ix^\rho P_\rho}\phi(0) = e^{-ix^\rho P_\rho}\left(e^{ix^\rho P_\rho}L_{\mu\nu}e^{-ix^\rho P_\rho}\right)\phi(0) \\ &= e^{-ix^\rho P_\rho}\left(L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu\right)\phi(0) \\ &= \left(S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu\right)\phi(x) \end{aligned} \tag{2.22}$$

This yields the action of Lorentz transformations at $x \neq 0$. Similarly, one can derive the following action of dilatations and SCTs on $\phi(x)$ (that of P_μ is clear since it simply generates a translation):

$$\begin{cases} D\phi(x) = \left(x^\mu P_\mu - i\Delta\right)\phi(x) \\ K_\mu\psi(x) = \left(2x_\mu x^\lambda P_\lambda - |x|^2 P_\mu - 2i\Delta x_\mu - 2x^\lambda S_{\lambda\mu}\right)\phi(x) \end{cases} \tag{2.23}$$

For a spinless field, which is the case here, we in fact have $S_{\mu\nu} = 0$ and the behavior of the field under the conformal group is essentially characterized by its scaling dimension Δ. From the above, one can derive the action of a finite conformal transformation on a scalar field ϕ:

$$\phi(x) \to \phi'(x') = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{-\Delta/d}\phi(x) \tag{2.24}$$

A field satisfying this transformation property is called a *quasi-primary field.* Indeed, all primaries satisfy this property, while quasi-primaries are not primary *a priori.* Being primary or quasi-primary is a heavy constraint, which can be used to drastically simplify the form of their correlation functions, while one can obtain information on descendants using the action of conformal generators on primaries.

2.4 Constraints on correlation functions

It turns out that conformal symmetry heavily constrains the correlation functions of a given theory. Let's consider a theory with a given action $S[\Phi]$ depending on a field configuration Φ. The path integral formulation allows one to write correlation functions in the following way:

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{1}{\mathcal{Z}}\int D[\Phi]\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)e^{-S[\Phi]} \tag{2.25}$$

where $\mathcal{Z}$ is a normalization factor, called the partition function by analogy with statistical mechanics. This combined with the transformation rule for primary operators entails the following transformation rule for correlation functions:

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \left|\frac{\partial \tilde{x}}{\partial x}\right|_{x=x_1}^{\Delta_1/d}\ldots\left|\frac{\partial \tilde{x}}{\partial x}\right|_{x=x_n}^{\Delta_n/d}\langle \mathcal{O}_1(x_1')\ldots\mathcal{O}_n(x_n')\rangle \tag{2.26}$$

In particular, let's consider this relation for a scale transformation $x \mapsto \lambda x$ and a two-point correlation function:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\rangle \tag{2.27}$$

Poincaré invariance constrains the two-point correlation function to depend only on the distance between two points:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f\left(|x_1 - x_2|\right) \tag{2.28}$$

The first constraint imposes the following functional equation on f:

$$f(x) = \lambda^{\Delta_1+\Delta_2}f(\lambda x) \tag{2.29}$$

Hence:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{f(1)}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{2.30}$$

where $f(1)$ is a constant depending on the two operators. It turns out that further restrictions can be derived using other conformal transformations. Indeed, invariance under finite SCTs, followed by a normalization condition which we do not explain here yields the following form [9]:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_{ij}}} \tag{2.31}$$

It is worth noting that this is exactly the dependence of the Ising model's two-point correlation function at its critical point, and the scaling dimension of the Ising spin field is directly related to the critical exponent η. As mentioned previously, at the critical point, the theory becomes scale invariant, which is a hallmark of conformal invariance.

Similarly, one can show that conformal invariance leads to the following form for three-point correlation functions:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{|x_1 - x_2|^{h_{123}}\,|x_1 - x_3|^{h_{131}}\,|x_2 - x_3|^{h_{231}}} \tag{2.32}$$

where $h_{ijk} = \Delta_i + \Delta_j - \Delta_k$. The values of Δ_i and λ_{ijk}, referred to as *CFT data*, are all that is needed in a CFT to compute any local observable (typically higher-order correlation functions). This follows from the *operator product expansion*, or OPE for short.

2.5 The Operator Product Expansion in CFT

Simply put, an OPE tells us that we can expand the product of two local operators in terms of local operators. Indeed, let $\mathcal{O}_i(x), \mathcal{O}_j(y)$ be two local operators. Then there exists a family of functions f_{ijk} (called structure constants) defined on some open set $\mathcal{U}$, such that for $z \in \mathcal{U}$:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k f_{ijk}(z)\mathcal{O}_k(z) \tag{2.33}$$

where the $\mathcal{O}_k$ comprise all of the operators of the theory. A typical situation where OPEs arise is when two local operators are inserted close together in a correlation function, and this is the situation which will arise in the next sections. In CFT, such an OPE takes the following form [10]:

$$\mathcal{O}_i(x)\mathcal{O}_j(y) \underset{x\to y}{=} \sum_k \frac{\lambda_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(x-y) \tag{2.34}$$

To determine the λ coefficients, one can look at the following three-point function and insert the OPE ansatz when $x \to 0$:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\mathcal{O}_k(y)\rangle = \sum_l \frac{\lambda_{ijl}}{|x|^{\Delta_i+\Delta_j-\Delta_l}}\langle \mathcal{O}_l(x)\mathcal{O}_k(y)\rangle \tag{2.35}$$

The operators appearing in the OPE can be primaries or descendants. In the latter case, this will amount to a conformal symmetry generator (typically a differential operator) applied to a primary, and hence the 2-point functions in the OPE can always be related to two-point functions of primaries. Recall that from CFT the two-point function of two primaries vanishes unless the two operators coincide. Hence, the three-point function simplifies drastically:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\mathcal{O}_k(y)\rangle = \frac{\lambda_{ijk}}{|x|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{D}\langle \mathcal{O}_k(x)\mathcal{O}_k(y)\rangle \tag{2.36}$$

where $\mathcal{D}$ is some differential operator. Let's suppose for the sake of simplicity that such differential operators don't appear. Using the expression of a (normalized) two-point function in CFT one then has for $x \to 0$:

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\mathcal{O}_k(y)\rangle = \frac{\lambda_{ijk}}{|x|^{\Delta_i+\Delta_j-\Delta_k}|y|^{2\Delta_k}} \tag{2.37}$$

One can then check that this is consistent with the expression of three-point functions in a CFT (the condition $x \to 0$ is crucial for this to be true), and hence one can relate the OPE coefficients to the CFT data. The OPE allows one to reduce the order of correlation functions to solve for, and hence the CFT data is enough to compute correlation functions of local observables to arbitrarily high order, as previously alluded to.

It is worth noting that OPEs are at the basis of conformal bootstrap methods [8, 11]. Consider a four-point correlation function $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle$. One can compute the OPE of $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$, or $\mathcal{O}_1\mathcal{O}_4$ and $\mathcal{O}_2\mathcal{O}_3$. The two expansions must coincide on the intersection of the associated sets of convergence; this yields a *crossing relation*. Imposing crossing relations on all four-point functions would significantly constrain the possible values of CFT data, although in practice one can only impose them on a smaller subset of correlation functions. The conformal bootstrap is one of the main fields of research related to CFT, but it lies beyond the scope of this work.

2.6 Stress tensors and a criterion for conformal invariance

As with any symmetry, the associated currents are important objects in a given theory. Among these currents, the stress tensor is an object of note in CFT. The canonical stress tensor of a field theory can be derived using spacetime translation symmetry. Suppose a scalar theory is described by the following action:

$$S = \int d^d x \mathcal{L} \left(\phi, \partial_\mu \phi\right) \tag{2.38}$$

Let's also suppose this action is invariant under the infinitesimal transformation $x \mapsto x + \varepsilon$, $\phi(x) \mapsto \phi(x) - \varepsilon^\mu \partial_\mu \phi(x)$. Taking into account the Euler-Lagrange equations yields the following variation of the action:

$$\delta S = \int d^d x \delta \mathcal{L} = \varepsilon^\mu \int d^d x \left(-\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} \right) \partial_\mu \phi - \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \phi\right)} \partial_\nu \partial_\mu \phi + \partial_\mu \mathcal{L} \right) \tag{2.39}$$

This is valid for all ε, hence $\partial_\mu T^{\mu\nu} = 0$ with

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{2.40}$$

This is often referred to as the *canonical* energy-momentum, or stress tensor. This tensor is not symmetric *a priori*. Nevertheless, it can be made symmetric by adding a divergence of a rank-3 tensor to the Lagrangian density (this yields the so-called Belinfante tensor if we use the current associated with rotational/Lorentz invariance, see [3] for example). Hence one usually assumes it is symmetric to begin with.

There is another way to derive and define the stress tensor, which renders it manifestly symmetric and yields a powerful criterion for conformal invariance. To motivate this, suppose a theory has a symmetric stress tensor. Then the variation of its action under $x \mapsto x + \varepsilon(x)$ to first order in ε is given by

$$\begin{aligned} \delta S &= -\int d^d x \varepsilon^\nu \partial_\mu T^{\mu\nu} \\ &= \int d^d x T^{\mu\nu} \partial_\mu \varepsilon^\nu \\ &= \frac{1}{2} \int d^d x T^{\mu\nu} \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) \end{aligned} \tag{2.41}$$

This expression can be rewritten using the variation of the metric induced by the change of coordinates. Indeed, to first order in ε, $g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu = g_{\mu\nu} + \delta g_{\mu\nu}$. Hence

$$\delta S = -\frac{1}{2} \int d^d x T^{\mu\nu} \delta g_{\mu\nu} \tag{2.42}$$

from which we derive the alternate definition of the stress tensor, using a functional derivative of the action with respect to the metric:

$$T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}} \tag{2.43}$$

It's worth noting that this definition coincides with the one from general relativity (in flat space), if one takes S to be the matter contribution to the Einstein-Hilbert action. Let's use the expression of $\delta g_{\mu\nu}$ in the case of conformal transformations. Indeed, one has $\delta g_{\mu\nu} = \kappa \eta_{\mu\nu}$. Plugging this into equation (2.42):

$$\delta S \propto \int d^d x \kappa T^\mu{}_\mu \tag{2.44}$$

We arrive at a very useful statement: a theory with a traceless stress tensor is conformal (this yields a fast check of conformal invariance for sourceless electrodynamics for example). This is very powerful, in that a calculation of the trace of an often relatively accessible tensor is much simpler than checking for conformal invariance under each family of conformal transformations. However, not all theories have stress tensors: the long-range Ising model, whose study we turn to in the last section, does not have one. Justifying conformal invariance in the LRI fixed point will prove non-trivial, and will require essential ingredients from QFT, which we turn to shortly.

2.7 Ward identities

To conclude this section on CFT, we derive another way of expressing conformal invariance following [3], which will yield the criterion we'll use to prove conformal invariance in the LRI fixed point. Consider the transformation of a field ϕ under the infinitesimal action of a generator G_a:

$$\phi'(x) = (1 - i\omega_a G_a)\phi(x) \tag{2.45}$$

In the path integral formalism, the n-point correlation function of ϕ, which we denote $\langle X \rangle$, can be written as

$$\langle X \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi X e^{-S[\Phi]} \tag{2.46}$$

If we change the functional integration variable according to equation (2.45) (the detailed derivations can be found in [3]):

$$\langle X \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi' (X + \delta X) e^{-S[\Phi] + \int dx \partial_\mu j_a^\mu \omega_a(x)} \tag{2.47}$$

where we use the current j_a^μ associated to the transformation in equation (2.45). Expanding this equation to first order in ω_a:

$$\langle \delta X \rangle = \int dx \partial_\mu \langle j_a^\mu(x) X \rangle \omega_a(x) \tag{2.48}$$

Next, the variation in X can be expanded explicitly:

$$\delta X = -i \int dx \omega_a(x) \sum_{i=1}^{n} \{\phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n)\} \delta(x - x_i) \tag{2.49}$$

Combining the last two equations above (which hold for arbitrary small ω_a), we get the local expression of the *Ward identity* for the current j_a^μ:

$$\partial_\mu \langle j_a^\mu \phi(x_1) \dots \phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \rangle \tag{2.50}$$

This equation provides a check that j_a^μ is a conserved current, since correlation functions effectively comprise all of the observables of the theory. In the case of CFT, the currents are the following (derivations follow the same line of reasoning as in the case of translations and the stress tensor, and again we can modify them to simplify their expressions by adding total derivatives to the action):

$$\begin{cases} j_D^\mu = T^{\mu\nu} x_\nu \\ j_K^{\mu\nu} = T^{\mu\rho} \left(2x_\rho x^\nu - \delta_\rho^\mu x^2\right) \\ j_L^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu \end{cases} \tag{2.51}$$

Furthermore, the LHS vanishes in equation (2.50) if we integrate over a large enough volume (using the divergence theorem and supposing cancelation on the boundary, or even requiring that the vacuum be conformally invariant in operator formalism [8]) and we are left with

$$\sum_{i=1}^{n} \langle \phi(x_1) \dots G_a \phi(x_i) \dots \phi(x_n) \rangle = 0 \tag{2.52}$$

This is the global Ward identity, which signals conservation of *charge*, rather than of current (the local conservation law implies the global one, not the other way around). We shall see that the LRI does not have a stress tensor, and hence we'll adopt a weaker definition: a theory is taken to be conformally invariant when the global conformal Ward identities hold: this will be essential in section 4.6. However, we first must turn to QFT to develop the necessary tools for explicitly calculating correlation functions.

3 Quantum Field Theory Essentials

3.1 Feynman rules

From here on out, we exclusively work in d-dimensional Euclidean space endowed with a metric δ_{ij}. As in quantum mechanics, computing expectation values of observables – which are represented by operators – is an essential part of QFT. The main operators in QFT are field operators, and as such solving a theory essentially means being able to compute any correlation function of the form $\langle\phi_1(x_1)\dots\phi_n(x_n)\rangle$ for a given field ϕ. Using the (Euclidean) path integral formulation, such a correlation function can be written as

$$\langle\phi_1(x_1)\dots\phi_n(x_n)\rangle = \frac{1}{Z_0}\int \mathcal{D}\Phi\phi_1(x_1)\dots\phi_n(x_n)e^{-S[\Phi]} \tag{3.1}$$

where Φ is a shorthand for all fields ϕ_i of the theory. This expression is extremely hard to compute exactly in the general case. Incidentally, it involves a measure over the space of all fields, which isn't well defined in the general case (a first approach might be to start with a measure on the lattice and take the continuum limit, but then one has to rigorously prove convergence). The standard way of computing such an expression is perturbation theory, treating the interaction part's coupling as small and using a diagrammatic expansion – known as Feynman diagrams – to compute successive orders of the expansion.

Suppose the action is written as a free part S_0 perturbed by a ϕ^4 interaction (this is the case we will typically consider when studying the LRI). Generally, a free action shall be written as

$$S_0 = \frac{1}{2}\int d^dxd^dy\phi(x)D(x,y)\phi(y) \tag{3.2}$$

where $D(x,y)$ is some functional operator (for the usual free field, or Klein-Gordon theory, one can take $D(x,y) = \delta(x-y)\Box_x$). The only case where exactly computing correlation functions of a field theory is feasible is that of *Gaussian* theories (see [12] for a discussion on Gaussian integrals in QFT). This is the case of the theory whose action is given by (3.2) above, since D can be viewed as the inverse of a functional equivalent of the covariance matrix, and thus S_0 can be interpreted as the exponent in the probability density of a centered normal distribution.

In order to calculate n-point correlation functions of the free theory, we make use of *Wick's theorem*. It stems from the fact that a two-point correlation function is given by the propagator $G(x-y) := D^{-1}(x,y)$, and states that for larger n, n-point correlation functions can be computed using sums and products of propagators (see appendix A for justification):

$$G^{(n)}(x_1,\dots,x_n) = \sum_{\text{all pairings}} G(x_{i_1}-x_{i_2})\dots G(x_{i_{n-1}}-x_{i_n}) \tag{3.3}$$

Incidentally, there are $n!/(2^{n/2}(n/2)!)$ such pairings, which is a number which blows up at large n. Wick's theorem can also be used for interacting theories treated perturbatively in λ. Indeed, supposing we perturb the Gaussian theory with a ϕ^4 term:

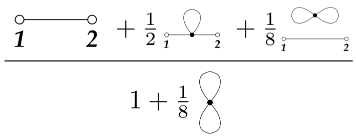
$$\begin{aligned}\langle\phi(x_1)\dots\phi(x_n)\rangle &= \frac{\langle\phi(x_1)\dots\phi(x_n)e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0}{\langle e^{-\frac{\lambda}{4!}\int d^dx\phi^4}\rangle_0} \\ &= \frac{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^kk!}\int d^dy_1\dots d^dy_k\langle\phi(x_1)\dots\phi(x_n)\phi^4(y_1)\dots\phi^4(y_k)\rangle_0}{\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{4!^kk!}\int d^dy_1\dots d^dy_k\langle\phi^4(y_1)\dots\phi^4(y_k)\rangle_0}\end{aligned} \tag{3.4}$$

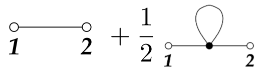
Note that we've swapped integral and summation signs without any rigorous justification. In general, these are not convergent series, but rather asymptotic series i.e. despite initially being convergent, they can diverge eventually. Wick's theorem then applies to the correlators with respect to the free theory. Each term in this expansion can be represented by a so-called *Feynman diagram*. Using notations from the expansion above, a given diagram has k internal vertices – signaling integration over position – and n external vertices, while propagators are represented by solid lines connecting vertices. In particular, for ϕ^4 theory, internal vertices have four "legs". These guidelines are referred to as *position-space Feynman rules*, and we will mainly make use of these in this work.

However, it is often practical to express everything in momentum space, for example for scattering amplitude calculations. This allows one to interpret a diagram's external legs as incoming or outgoing particles of given momenta, the internal vertices being particle interactions. Mathematically, this essentially entails keeping the propagator in momentum space and using the correspondence between convolutions and products under Fourier transform. External lines then carry a given momentum, internal vertices come with a coupling factor (like for position-space calculations) with the added condition of conservation of momentum (coming from δ-functions via integration of complex exponentials). Any internal momenta are then integrated over. These guidelines are referred to as (momentum-space) *Feynman rules*, and we will briefly use them in section 3.3.

3.2 A first example: two-point correlation function

As a quick exercise, let's express the two-point correlation function of ϕ^4 theory to first order in perturbation theory.

$\langle \phi(x_1)\phi(x_2)\rangle =$  $+ \mathcal{O}(\lambda^2)$

$=$  $+ \mathcal{O}(\lambda^2)$ (3.5)

$$= G(x_1 - x_2) - \frac{\lambda}{2}\int dy G(y-x_1)G(0)G(y-x_2) + \mathcal{O}(\lambda^2)$$

It is worth noting that dividing by the partition function has simplified the term multiplied by 1/8. Such a diagram is called "disconnected" – the others being connected diagrams – since it contains an internal vertex which is disconnected from all external vertices. Generally, division by the partition function will rid the expansion of disconnected diagrams, although we do not rigorously prove this here.

Furthermore, the first order correction to this correlation function is what is often referred to as a *tadpole* diagram. Diagrams such as these involve propagators computed at zero spacing, which are usually divergent. In the case of a CFT for example, a two-point function goes like $|x-y|^{-2\Delta_\phi}$, which generally diverges at $x = y$. Getting rid of divergences such as these is one of the reasons for renormalization.

3.3 One-loop renormalization of ϕ^4 theory

As an example, let's consider Klein-Gordon theory perturbed by a ϕ^4 interaction:

$$S = \int d^dx \left(\partial_\mu\phi\partial^\mu\phi + \frac{m_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4\right) \tag{3.6}$$

Its (Euclidean) propagator is given by:

$$G(x-y) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int d^dp \frac{e^{ip(x-y)}}{p^2+m_0^2} \tag{3.7}$$

When it is evaluated at $x = y$, using spherical coordinates (see details in [13, 14]):

$$\begin{aligned} G(0) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int \frac{d^dp}{p^2+m_0^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Gamma\left(\frac{d}{2}\right)}\int_0^{+\infty} dp \frac{p^{d-1}}{p^2+m_0^2} \\ &= \frac{\left(m_0^2\right)^{d/2-1}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right) \end{aligned} \tag{3.8}$$

If we work in 4 dimensions, this quantity diverges. For the remainder of this work, we adopt the *dimensional regularization scheme*, which focuses on getting rid of divergences caused by the dimension of space by perturbing it. Let's define $\varepsilon = 4 - d \ll 1$.

$$G(0) = \frac{\left(m_0^2\right)^{1-\frac{\varepsilon}{2}}}{(4\pi)^{2-\frac{\varepsilon}{2}}}\Gamma\left(\frac{\varepsilon}{2}-1\right) \tag{3.9}$$

In momentum space, the previous tadpole diagram yields a contribution

$$-\frac{1}{2}m_0^2\frac{\lambda_0}{(4\pi)^2}\left(\frac{4\pi}{m^2}\right)^{\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}-1\right) \tag{3.10}$$

We introduce a dimensionless coupling g such that $g_0 = g\mu^\varepsilon$ (one can easily check using dimensional analysis that $[g_0] = L^{d-4}$) where μ is an energy scale. The tadpole contribution can now be written

$$-\frac{1}{2}m^2 \frac{g}{(4\pi\mu^2)^2}\left(\frac{4\pi}{m^2}\right)^{\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}-1\right) \underset{\varepsilon\to 0}{=} \frac{1}{2}m^2\frac{g}{(4\pi)^2}\left(\frac{2}{\varepsilon}+\psi(2)+\ln\left(\frac{4\pi\mu}{m^2}\right)+\mathcal{O}\left(\varepsilon^2\right)\right) \tag{3.11}$$

We don't spend time on justifying this asymptotic expansion here, further details can be found in [13]. This expression yields a first-order pole in ε. One trick for getting rid of it is to use a counter-term in the original action. This effectively consists in rewriting the action with respect to the *renormalized* quantities $[\phi]$, $[\phi^2]$, $[\phi^4]$ m, λ, as opposed to the *bare* quantities ϕ, m_0, λ_0.

$$S = \int d^d x \left(\frac{1}{2}\partial_\mu\left(Z_1^{\frac{1}{2}}[\phi]\right)\partial^\mu\left(Z_1^{\frac{1}{2}}[\phi]\right) + \frac{Z_m m^2}{2}Z_2[\phi^2] + \frac{Z_\lambda \lambda}{4!}Z_4[\phi^4]\right) \tag{3.12}$$

where for any index A:

$$Z_A = 1 + \sum_{k=1}^{+\infty}\frac{f_k^A(g)}{\varepsilon^k} \tag{3.13}$$

and $f_k(g) = \mathcal{O}(g^k)$. The constant terms account for the original action, while the $k = 1$ terms are the so-called one-loop counter-terms added to the Lagrangian to get rid of the poles appearing in the tadpole diagram as well as other diagrams. Moreover, Z factors in front of fields are typically referred to as *wave-function renormalization* factors. The wavefunction and mass renormalization modifies the momentum-space propagator to first order in g [15]:

$$\begin{aligned}\hat{G} &= \frac{1}{Z_1 p^2 + Z_m Z_2 m^2} \approx \frac{1}{p^2 + m^2 + \frac{f_1^1(g)p^2 + (f_1^m(g)+f_1^2(g))m^2}{\varepsilon}} \\ &\approx \frac{1}{p^2+m^2} - \frac{1}{\varepsilon}\frac{1}{p^2+m^2}\left(f_1^1(g)\,p^2 + (f_1^m(g)+f_1^2(g))m^2\right)\frac{1}{p^2+m^2}\end{aligned} \tag{3.14}$$

The second term yields an interaction term $-\left(f_1^1(g)\,p^2 + \left(f_1^m(g)+f_1^2(g)\right)m^2\right)/\varepsilon$ in momentum space, which we want to counteract the tadpole interaction exactly. This can be done for $f_1^1(g) = 0, f_1^2(g) = 0$ and

$$f_1^m(g) = \frac{g}{16\pi^2} \tag{3.15}$$

This leads to

$$m_0^2 = m^2\left(1 + \frac{g}{16\pi^2}\frac{1}{\varepsilon}\right) \tag{3.16}$$

where m can be interpreted as an effective mass at the scale μ, accounting for all of the one-loop interactions. It is worth noting that in this scheme, we've only gotten rid of the divergent term in the tadpole diagram, and we've left the finite terms intact. This is called the *minimal subtraction*, or MS scheme. We could also get rid of the finite terms in the "extended'" minimal subtraction, or $\overline{\text{MS}}$ scheme. Furthermore, notice that in the one-loop structure of ϕ^4 theory, Z_1 and $Z_2 = 1$. This implies that the renormalized and the bare fields coincide (in section 3.5 we'll also learn to say that the field's anomalous dimension is zero).

In one-loop renormalization, tadpoles are not the only diverging class of diagrams. Diagrams in the second-order expansion of the four-point correlation function also lead to divergences, which can be removed by acting on Z_λ. It is convenient to reason in momentum space here as well. For incoming total momentum k, modulo permutations, the divergent integral is

$$\int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2+m^2)\left[(k-p)^2+m^2\right]} \supset \mu^\varepsilon \frac{g^2}{(4\pi)^2}\frac{2}{\varepsilon} \tag{3.17}$$

where $\supset$ denotes the inclusion of a pole in the expansion of the integral (see [13] for justification). There are three diagrams which contribute such poles, and a combinatorial factor to take into account. This leads to

$$\lambda_1 = \lambda\left(1 + \frac{3g\mu^{-\varepsilon}}{(4\pi)^2}\frac{1}{\varepsilon}\right) \tag{3.18}$$

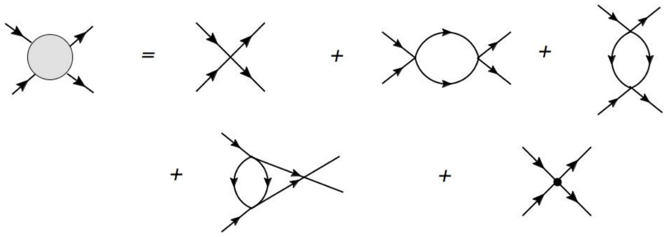


Figure 1: Expansion of the four-point function up to second order in the coupling in momentum space (fixed incoming and outgoing momenta). Taken from [16].

3.4 Another point of view: the renormalization group

It might seem difficult to attribute physical meaning to the previously described MS scheme beyond its mathematical utility. However, one can resort to another picture to develop a better intuition of what is going on in the previous subsection, namely that of the *renormalization group*. Broadly speaking, the renormalization group stems from the observation that, in order to describe a physical system with many degrees of freedom *at a given scale*, one can forget the microscopic details at much smaller scales. Increasing the scale leads to an effective "coarse-grained" theory with scale-dependent couplings, and this scale-dependence is constrained by the requirement that the physics remain unchanged under these so-called renormalization transformations. Incidentally, these transformations form a semi-group under composition, hence the looser denomination "renormalization group" (RG for short).

We therefore equip a given theory with a maximum energy scale (or momentum scale in what follows, the distinction is unimportant here), which would incidentally provide a regularization scheme to make some of the integrals in the previous section finite. Let's start off with a maximum energy scale μ_0. Let C_μ be the set of scalar fields whose energy is less than μ, and C_{μ_1,μ_2} the set of fields whose energy is between μ_1 and μ_2. For the sake of illustration, suppose we start with the following action:

$$S[\phi] = \int d^d x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_i g_{i0} \mathcal{O}_i(x) \right) = S_0[\phi] + S_1[g_{i0}; \phi] \tag{3.19}$$

where S_0 denotes the free field action, S_1 denotes the interaction part, the g_{i0} are the bare coupling constants, and $\mathcal{O}_i$ are local operators, typically powers of the ϕ fields. A regularized partition function is then given by:

$$\mathcal{Z}_{\mu_0} = \int_{C_{\mu_0}} \mathcal{D}\phi e^{-S[g_{i0};\phi]} \tag{3.20}$$

Next, we split a given field into its low energy and high energy parts relative to a new scale $\mu < \mu_0$, respectively χ and ψ. This can be done by introducing the cutoffs into the inverse Fourier transform. We thus have $\phi = \chi + \psi$. Explicitly writing the action with respect to this decomposition yields the following decomposition of the action:

$$S[\phi] = S^0[\chi] + S^0[\psi] + S_{\text{int}}[\chi, \psi] \tag{3.21}$$

where we've also assumed that the local operators have the appropriate form to lead to the above expression. We also assume that the path integral measure factorizes appropriately.

$$\mathcal{Z}_{\mu_0} = \int_{C_\mu} \mathcal{D}\chi e^{-S^0[\chi]} \int_{C_{\mu,\mu_0}} \mathcal{D}\psi e^{-S^0[\psi] - S_{\text{int}}[\chi,\psi]} \tag{3.22}$$

Here, the scale transformation consists in going from an integral over C_{μ_0} to one over C_μ. Thus, we can define a new action for the interactions, which describes the effective interactions at the new scale μ:

$$S_{\text{eff},\mu}[\chi] = -\ln \left[\int_{C_{\mu,\mu_0}} \mathcal{D}\psi e^{-S^0[\psi] - S_{\text{int}}[g_{i0};\chi,\psi]} \right] \tag{3.23}$$

By construction, this renormalization transformation preserves the partition function $\mathcal{Z}_\mu$, where we introduce its dependence on the scale μ. One can then introduce new, "running" couplings $g_i(\mu)$ describing the effective interactions depending on the μ scale. Supposing that the partition function depends smoothly on the scale, taking the derivative of the equation expressing the invariance of the partition function with respect to μ yields:

$$\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial g_i}{\partial\mu}\frac{\partial}{\partial g_i}\right]\mathcal{Z}_\mu(g)=0 \tag{3.24}$$

This equation is what is known as a *Callan-Symanzik* equation. Note that we've added a factor of μ. This is because conventionally one uses the logarithm of the UV cutoff (another expression for μ, ultraviolet signaling high energy). The above Callan-Symanzik equation then becomes:

$$\left[\frac{\partial}{\partial\ln\mu}+\beta_i(g_i)\frac{\partial}{\partial g_i}\right]\mathcal{Z}_\mu(g)=0 \tag{3.25}$$

where we define a new quantity – the β-function for a coupling g – which tells us how the coupling depends on the energy scale:

$$\beta(g):=\frac{\partial g}{\partial\ln\mu} \tag{3.26}$$

This function gives a very useful criterion for determining whether or not a point in the space of theories is a critical fixed point. Indeed, for such a point of coupling g_*, $\beta(g_*)=0$. This will be essential to defining a fixed point of the LRI in the last section.

3.5 Callan-Symanzik equations and anomalous dimensions

Let's keep the new RG picture. Suppose that under a renormalization flow, the effective action can be expressed as a bare action with counterterms via renormalization parameters like $Z(g,\varepsilon)$. At a scale μ, the n-point correlation function for a local operator $\mathcal{O}(x)$ can be written as:

$$G_\mu(x_1,\dots,x_n)=\frac{1}{\mathcal{Z}_\mu}\int_{C_\mu}\mathcal{D}\Phi\mathcal{O}(x_1)\dots\mathcal{O}(x_1)e^{-S_{\text{eff},\mu}[\Phi]} \tag{3.27}$$

The Z factors, previously used as a "mathematical trick" to subtract infinities, can be interpreted as the consequence of quantum corrections of the action dude to the high energy modes being integrated out. Clearly, since the change in the integration measure in the numerator cancels with that in the denominator, the n-point correlation function for bare operators is related to the one for running operators via:

$$G_0(x_1,\dots,x_n)=Z^nG_\mu(x_1,\dots,x_n) \tag{3.28}$$

Requiring the non-renormalized correlator to remain scale independent leads to the following generalized Callan-Symanzik equation:

$$\left[\frac{\partial}{\partial\ln\mu}+\beta_i\frac{\partial}{\partial g_i}+n\frac{1}{Z}\frac{\partial Z}{\partial\ln\mu}\right]G_\mu(x_1,\dots,x_n)=0 \tag{3.29}$$

It's worth noting that the third term depends on the power of Z (we used $\phi=Z^{1/2}[\phi]$ for the Klein-Gordon field), and hence there might occasionally be an additional factor of 1/2 in the literature. This equation deviates from the Callan-Symanzik equation of the previous subsection by a term proportional to

$$\gamma_{\mathcal{O}}:=\frac{1}{Z}\frac{\partial Z}{\partial\ln\mu} \tag{3.30}$$

This is known as the *anomalous dimension* of the field. Incidentally, this is to be compared to the scaling dimension of an operator. Indeed, let $s=\mu|x|$ be a dimensionless variable such that

$$G(x,0)=\frac{f(s,g)}{|x|^{2\Delta_{\mathcal{O}}}} \tag{3.31}$$

where we use the expression of a two-point function in CFT to provide this ansatz. At large distances, or in the IR (infrared) as it is often said in the literature, scale dependence vanishes and the Callan-Symanzik equation states that $[s\partial_s+2\gamma_{\mathcal{O}}]\,f(s,g)=0$. Integrating this and plugging it back into $G(x,0)$ yields

$$G(x,0)\propto\frac{1}{|x|^{2(\Delta_{\mathcal{O}}+\gamma_{\mathcal{O}})}} \tag{3.32}$$

Hence, $\gamma_{\mathcal{O}}$ is referred to as an anomalous dimension in that it encodes how the scaling dimension shifts from its classical scaling dimension under an RG flow [17]. Note that we've previously established that $Z_1=1$ under one-loop renormalization, and therefore the Klein-Gordon field didn't gain an anomalous dimension.

3.6 ϕ^4 theoretic β-function and Wilson-Fisher fixed point

Let's return to the one-loop renormalization of ϕ^4 theory. The renormalization of the λ coupling involves introducing a factor Z such that:

$$\lambda_0 = Z(g,\varepsilon) g \mu^{\varepsilon} \tag{3.33}$$

where $Z(g,\varepsilon) = 1 + f(g)/\varepsilon$ for one-loop renormalization, and $f(g)$ is a power series in g which is a $\mathcal{O}(g)$. The bare λ coupling must be independent of the μ scale, hence

$$\frac{\partial \lambda_0}{\partial \ln \mu} = \beta(g)\,\mu^{\varepsilon}\left(1+\frac{f}{\varepsilon}\right) + g\varepsilon\mu^{\varepsilon}\left(1+\frac{f(g)}{\varepsilon}\right) + \frac{g\mu^{\varepsilon} f'(g)}{\varepsilon}\beta(g) = 0 \tag{3.34}$$

i.e. to second order in g:

$$\beta(g) = -\varepsilon g + g^2 f'(g) \tag{3.35}$$

For ϕ^4 theory, we've already established that $f(g) = \frac{3g}{16\pi^2}$. Before going further, it will be useful to introduce some nomenclature: an interaction is said to be: *relevant* when it becomes significant at low energies (or its coupling has dimensions of a positive power of energy); *irrelevant* when it becomes insignificant at low energies (or its coupling has dimensions of a negative power of energy) and marginal otherwise (when the coupling is dimensionless). Notice that there is a trivial fixed point given by $g = 0$. This is just the free theory (Klein-Gordon here). Since the coupling is relevant, the interaction becomes negligible at high energies, and therefore one refers to this as the ultraviolet (UV) fixed point. However, there is a non-trivial fixed point given by [17]:

$$g = g_* = \frac{\varepsilon}{f'(g)} = \frac{16\pi^2\varepsilon}{3} \tag{3.36}$$

This is sometimes referred to as the Wilson-Fisher (WF) fixed point, and it is an IR fixed point. Indeed, one typically drives a theory to the WF point by turning on a marginally relevant interaction, thus making it flow to the IR. At this point, we recover scale invariance. However, this does not automatically imply conformal invariance. Nevertheless, we will be working at an analogous fixed point in section 4.6 and show that this scale invariance is indeed promoted to conformal invariance.

3.7 The OPE and normal-ordering

To conclude this section on the essential tools from QFT, let's look at another useful way of renormalizing a theory (typically a CFT here). Indeed, another way to regularize diagrams with zero-spacing propagators $G(0)$ (tadpoles) is normal-ordering. In canonical quantization, this amounts to keeping annihilation operators "on the right" relative to creation operators. Expressions are typically sandwiched between $\langle 0|$ and $|0\rangle$ since one usually evaluates vacuum expectations, and this ordering ensures many of the vacuum terms are annihilated, thus significantly simplifying calculations.

In the Gaussian theories considered in this work – which are also CFTs – normal-ordering a product of operators is closely linked to the OPE. Indeed, consider two local operators $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the following OPE holds for $x \to y$:

$$\mathcal{O}_1(x) \times \mathcal{O}_2(y) = \sum_{k=1}^{\infty} \frac{\lambda_{12k}}{|x-y|^{\Delta_1+\Delta_2-\Delta_k}} \mathcal{O}_k(x-y) \tag{3.37}$$

where we order the operators of the theory by increasing scaling dimension. Let $K \in \mathbb{N}^*$ such that $\Delta_k \geq \Delta_1 + \Delta_2$. Such a K always exists in a Gaussian theory, since one can increase scaling dimensions by taking arbitrarily high powers of a given operator (no anomalous dimensions). Therefore, when $x \to y$, the following quantity remains finite:

$$\begin{aligned} :\mathcal{O}_1(x) \times \mathcal{O}_2(y): &= \mathcal{O}_1(x) \times \mathcal{O}_2(y) - \sum_{k=1}^{K-1} \frac{\lambda_{12k}}{|x-y|^{\Delta_1+\Delta_2-\Delta_k}} \mathcal{O}_k(x-y) \\ &= \sum_{k=K}^{\infty} \frac{\lambda_{12k}}{|x-y|^{\Delta_1+\Delta_2-\Delta_k}} \mathcal{O}_k(x-y) \end{aligned} \tag{3.38}$$

This is referred to as the *normal-ordered product* of $\mathcal{O}_1$ with $\mathcal{O}_2$, and one can check by inserting it in correlation functions that this eliminates tadpole divergences. This is particularly useful in a massless theory (which has to be the case for a CFT), since one cannot conduct mass renormalization as we did in a previous subsection to take care of tadpoles. In what follows, we will always assume products to be normal-ordered and drop the double-colon notation. We are now fully equipped to address the LRI.

4 The Long-Range Ising Model

4.1 Introduction to the LRI

Spin models, such as the Ising model, were historically introduced to explain ferromagnetism, although their study is now useful to the understanding of entire universality classes. According to classical electrodynamics, a magnetic dipole $\vec{\mu}$ in d spatial dimensions produces a magnetic field given by

$$\vec{B}(\vec{r}) = \frac{k}{r^{d+2}} \left(d\,(\vec{r} \cdot \vec{\mu})\, \vec{r} - r^2 \vec{\mu} \right) \tag{4.1}$$

where k is a constant depending on the magnetic permeability of free space and the geometric properties of a $(d-1)$-sphere. Therefore, the potential energy of interaction between two dipoles $\vec{\mu}_1$, $\vec{\mu}_2$ is:

$$V = -\frac{k}{r^{d+2}} \left(d\,(\vec{r} \cdot \vec{\mu}_1)\,(\vec{r} \cdot \vec{\mu}_2) - r^2 (\vec{\mu}_1 \cdot \vec{\mu}_2) \right) \tag{4.2}$$

Hence, the interaction between dipoles goes like the product of dipole moments (which we take to be spins in accordance with the Wigner-Eckart theorem) times a term that goes like $1/r^d$. In the usual Ising model, we ignore spatial dependence and only keep nearest-neighbor interactions. However, we decide here to keep such dependence and introduce the long-range equivalent of the Ising model.

The LRI is a "non-local" generalization of the usual short-range Ising model (SRI). Indeed, in this model, we do not merely consider nearest-neighbor interactions. Rather, we consider that each spin interacts with every other spin via an interaction going like $1/r^{d+\sigma}$ where r is the distance between the two spins considered. The constant σ can be viewed as the discrepancy between the LRI and the spatial dependence of a dipole interaction from classical electrodynamics. More specifically, for a ferromagnetic system, we'll consider the following Hamiltonian:

$$H = -J \sum_{i \neq j} \frac{s_i s_j}{|i-j|^{d+\sigma}} \tag{4.3}$$

where one can use a Euclidean norm in d dimensions to define $|i-j|$ in the sum above. Depending on the value of σ, this model shall exhibit different behaviors. Clearly, for large σ, long-range interactions will be suppressed and we'll recover the SRI. However, it turns out that there are two other regimes, characterized by a Gaussian and non-Gaussian critical point respectively. In order to study the σ dependence of the LRI, we shall introduce a continuum theory which belongs to the same universality class.

4.2 The continuum theory

Let's consider the continuum limit of the LRI. This amounts to introducing a lattice spacing a and making it go to zero. We can then define a scalar field ϕ such that $\phi(ai) = s_i$. This leads to the following Hamiltonian:

$$H = -Ja^{\sigma-d} \sum_{i \neq j} \frac{\phi(ai)\phi(aj)}{|ai-aj|^{d+\sigma}} a^{2d} \tag{4.4}$$

Taking a coupling $J(a) = J_0 a^{d-\sigma}$ allows this quantity to behave like the following continuum limit as $a \to 0$:

$$H = -J(a) \int d^d x d^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} \tag{4.5}$$

Therefore, since one can flow from the lattice to the continuum, in accordance with universality, one can replace the study of the discrete model by a continuous theory. Moreover, we turn on a relevant ϕ^4 interaction since we'll want to drive the theory to the IR:

$$S = S_0 + S_1 = \int d^d x d^d y \frac{\phi(x)\phi(y)}{|x-y|^{d+\sigma}} + \frac{g_0}{4!} \int d^d x \phi(x)^4 \tag{4.6}$$

Note that we haven't included a mass term, quadratic in the scalar field. Indeed, such a term would introduce a length scale (length has units of inverse mass in natural units).

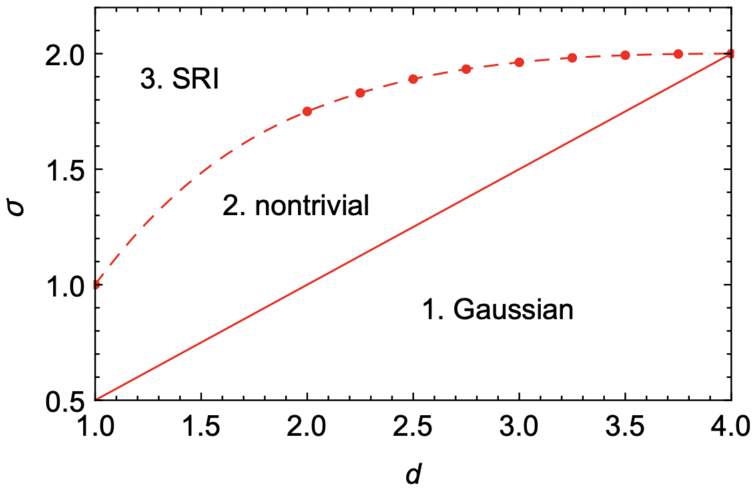


Figure 2: Critical behavior of the LRI according to different values of d and σ. Plot taken from [2].

It turns out that the first term can be defined using a "fractional Laplacian" $\mathcal{L}_\sigma = \left(-\partial^2\right)^{\sigma/2}$ (see appendix B.1 for detailed justification), and we can rewrite the full action as

$$S = S_0 + S_1 = \frac{\mathcal{N}_\sigma}{2} \int d^d x \phi(x) \mathcal{L}_\sigma \phi(x) + \frac{g_0}{4!} \int d^d x \phi(x)^4 \tag{4.7}$$

where we introduce a normalization factor $\mathcal{N}_\sigma$ which normalizes the two-point function of the free Gaussian theory (this constant doesn't affect the physics of the LRI). It's worth noting that the fractional Laplacian is a non-local operator, as opposed to the usual d'Alembertian. Indeed, it can be defined up to a normalization by:

$$\mathcal{L}_\sigma \phi(x) \propto \int d^d y \frac{\phi(y)}{|x-y|^{d+\sigma}} \tag{4.8}$$

The first term in this action closely resembles that of a free scalar field, if one allows for the $D(x,y)$ operator to be non-local (recall notation from section 3.1). This term is in fact Gaussian, hence a field whose action is S_0 is a type of *generalized free field* (GFF).

4.3 Critical behavior

Before delving into the discussion of conformal invariance in the LRI, let's start by giving a qualitative description of how the critical theory changes depending on the σ parameter. As a prerequisite, let's calculate the dimension of the coupling constant g_0 in equation (4.7). Clearly, the GFF has dimensions of $\mu^{(d-\sigma)/2}$, hence g_0 has dimensions of $\mu^{d-2\sigma}$ (where μ stands for an energy scale). This entails that the quartic interaction is irrelevant for $\sigma < d/2$, marginal for $\sigma = d/2$ and relevant for $\sigma > d/2$. It turns out that there is another boundary for σ, defined using a critical exponent from the SRI. Indeed, the scaling dimension of ϕ is $\Delta_\phi = (d-\sigma)/2$ (this can be easily shown using a change of variables in the free action, or simply by dimensional argument), and from the constraints of CFT on correlation functions $\langle \phi(x)\phi(y)\rangle = |x-y|^{\sigma-d}$ (we'll see that the GFF theory is conformally invariant shortly). In the SRI, one defines the η critical exponent, which depends on d, such that at the second order phase transition:

$$\langle s_i s_j \rangle = \frac{1}{|i-j|^{d+\eta_{SRI}-2}} \tag{4.9}$$

Hence, for $\sigma \geq \sigma^* := 2 - \eta_{SRI}$, one can expect the critical theory to be the SRI, and this is in fact the case. From the previous subsection, when the quartic perturbation is irrelevant i.e. when $\sigma < d/2$, one can take the critical theory to be that of a GFF, also referred to as a Gaussian theory. In summary, the critical theory is: Gaussian for $\sigma < d/2$, non-trivial, non-Gaussian for $d/2 < \sigma < \sigma^*$ and SRI for $\sigma^* < \sigma$.

It is also worth noting what happens at $d = 4$. This value is the upper critical dimension of the SRI, hence the corresponding critical exponents are those of the mean field: $\eta_{\text{SRI}} = 0$. The LRI fixed point is given by $\sigma = d/2 = 2$ while the SRI crossover happens at $\sigma = \sigma^* = 2$; these two values coincide and therefore the theory directly crosses over from a GFF to the SRI. In order to study the non-trivial theory, we shall therefore assume $d < 4$.

In the following subsections, we study the GFF + ϕ^4 theory given above in order to justify the conformal invariance of the LRI at the $d = \sigma/2$ crossover. Unfortunately, the LRI does not have a stress tensor (see [18] for a motivation of this fact), and as such justifying conformal invariance will require more involved arguments.

4.4 The Gaussian theory

In this section, we prove conformal invariance of the Gaussian/GFF theory and calculate some correlation functions using the machinery developed in section 3.

4.4.1 The propagator

As a prerequisite to using Wick's theorem to calculate correlators, we need to calculate a fundamental building block: the propagator of the GFF theory. It is given by the Green's function of the fractional Laplacian, which can be solved for using the Fourier transform:

$$G(x_1 - x_2) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot(x_1 - x_2)}}{|p|^\sigma} \tag{4.10}$$

If it is consistent with CFT, this quantity must go like $|x-y|^{-2\Delta_\phi}$. To show that this holds, we use the following identity (easily derived using a change of variables in the definition of the Γ-function):

$$\frac{1}{|p|^\sigma} = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)} \int_0^{+\infty} t^{\frac{\sigma}{2}-1} e^{-t|p|^2} dt \tag{4.11}$$

Next, we plug this into equation (4.10). This implies (integrals can be swapped using Fubini's theorem):

$$G(x) = \frac{1}{\Gamma\left(\frac{\sigma}{2}\right)} \int_0^{+\infty} dt t^{\frac{\sigma}{2}-1} \int \frac{d^d p}{(2\pi)^d} e^{-t|p|^2 + ipx} \tag{4.12}$$

The momentum space integral is a well-known Gaussian integral, whose evaluation yields:

$$G(x) = \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{\sigma}{2}\right)} \int_0^{+\infty} dt e^{-\frac{|x|^2}{4t}} t^{\frac{\sigma-d}{2}-1} \tag{4.13}$$

Next, the change of variables $u = |x|^2/(4t)$ turns the remaining integral into a Γ-function, and we obtain the desired result:

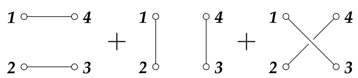
$$G(x) = \frac{2^{d-\sigma}\Gamma\left(\frac{d-\sigma}{2}\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{\sigma}{2}\right)} \frac{1}{|x|^{d-\sigma}} \tag{4.14}$$

Hence the propagator satisfies the conformality constraints, since $\Delta_\phi = (d-\sigma)/2$ is the scaling dimension of the field ϕ. We can then use the normalization factor $\mathcal{N}_\sigma$ in front of the GFF action to tune the propagator to the conventional form $G(x-y) = |x-y|^{-2\Delta_\phi}$. Incidentally, since all correlation functions are given by sums of products of correlation functions, this computation shows that the free Gaussian theory is conformally invariant (for a more direct argument, see appendix B.2).

4.4.2 Example of a three-point function

As a follow-up to the closing remark of the previous subsection, let's compute a three point function of the theory, for example $\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\rangle$. Wick's theorem tells us that:

$$\langle\phi(x_1)\phi(x_2)\phi^2(x_3)\rangle =$$



$$\begin{aligned} &= G(x_1 - x_3)G(x_2 - x_3) + G(x_1 - x_3)G(x_2 - x_3) \\ &= \frac{2}{|x_1 - x_3|^{2\Delta_\phi}|x_2 - x_3|^{2\Delta_\phi}} \end{aligned} \tag{4.15}$$

Note that we ignore divergent tadpole diagrams, since we take normal-ordered fields. It turns out that this respects the form of three-point functions in CFTs. Indeed, ϕ^2 doesn't gain an anomalous dimension

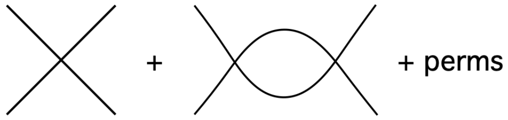


Figure 3: Diagrams in the expansion of the four-point correlation function requiring renormalization of the quartic coupling. Taken from [2].

in a Gaussian theory (this might be the case in an interacting theory however). Hence $\Delta_{\phi^2} = 2\Delta_\phi$, and $|x_1 - x_2|^{2\Delta_\phi - \Delta_{\phi^2}} = 1$. We can also extract the CFT datum $\lambda_{112} = 2$ here.

4.5 Renormalization of the GFF + ϕ^4 theory

Renormalization of the coupling. Let's consider the renormalization via dimensional regularization of the GFF theory perturbed by a marginally relevant ϕ^4 interaction. We will therefore be working at $\sigma = (d+\varepsilon)/2$ with $\varepsilon \ll 1$, and use a dimensionless coupling g such that $g_0 = g\mu^\varepsilon$ for an energy scale μ. Recall that we're working with normal-ordered products, hence we only need to renormalize the coupling by studying the four-point correlation function of ϕ. Indeed, divergences occur due to the diagrams in figure 3 present in the expansion of the four-point correlation function. These divergences arise when the ϕ^4 insertions are close to each other, thus the OPE $\phi^4(x) \times \phi^4(0)$ in the Gaussian theory proves useful in this situation (see appendix C for a derivation):

$$\phi^4(x) \times \phi^4(0) \supset \frac{72}{|x|^{d-\varepsilon}} \phi^4(0) \tag{4.16}$$

This entails that the second diagram in figure 3 yields a short-distance contribution

$$72 \times \frac{1}{2} \times \frac{g^2 \mu^{2\varepsilon}}{4!^2} \int_{|x| \ll 1} \frac{d^d x}{|x|^{d-\varepsilon}} = 36 \frac{g^2}{4!^2} \frac{S_d}{\varepsilon} + \text{finite term} \tag{4.17}$$

where S_d is the volume of the unit $(d-1)$-sphere (easily derived by expressing $\sqrt{\pi}^d$ as a d-dimensional Gaussian integral and going to spherical coordinates). In order to remove this pole, we write $g_0 = Z(g,\varepsilon)\mu^\varepsilon$, with

$$Z(g,\varepsilon) = 1 + \sum_{i=1}^{+\infty} \frac{f_k(g)}{\varepsilon^k} \tag{4.18}$$

In one-loop renormalization, one can therefore shift the coupling by

$$\delta g = \frac{36 S_d g^2}{4! \varepsilon} = \frac{3 S_d g^2}{2\varepsilon} \tag{4.19}$$

which corresponds to $f_1(g) = 3S_d g/2 = Kg$. This entails that the β-function of the theory up to $\mathcal{O}(g^3)$ is

$$\beta(g) = -\varepsilon g + K g^2 = -\varepsilon g + \frac{3 S_d g^2}{2} \tag{4.20}$$

The corresponding Wilson-Fisher critical point is then given by $g_* = \varepsilon/K$. The main goal of the final part of this work is to justify conformal invariance at this fixed point.

Wavefunction renormalization of ϕ^n. It turns out that composite operators ϕ^n need to be renormalized. We hence define the renormalized composite fields $[\phi^n]$ such that $\phi^n = Z_n(g,\varepsilon)\,[\phi^n]$. Consider the following two-point function:

$$\langle \phi^n(x)\phi^n(0) \rangle = \langle \phi^n(x)\phi^n(0) \rangle_0 - \frac{g\mu^\varepsilon}{4!} \int d^d y \langle \phi^n(x)\phi^n(0)\phi^4(y) \rangle_0 + \mathcal{O}(g^2) \tag{4.21}$$

where when the insertion $\phi^4(y)$ approaches $\phi^n(0)$, a pole in ε appears. Indeed, according to the OPE of $\phi^n(0) \times \phi^4(y)$ (again, see appendix C for a derivation):

$$\phi^n(x) \times \phi^4(0) \supset \frac{6n(n-1)}{|x|^{d-\varepsilon}} \phi^n(0) \tag{4.22}$$

which entails

$$\begin{aligned}\langle\phi^n(x)\phi^n(0)\rangle &\supset \langle\phi^n(x)\phi^n(0)\rangle_0\left(1-\frac{g\mu^\varepsilon}{4!}6n(n-1)\int_{|y|\ll 1}\frac{d^d y}{|y|^{d-\varepsilon}}\right)\\ &\supset \langle\phi^n(x)\phi^n(0)\rangle_0\left(1-\frac{n(n-1)S_d g}{4\varepsilon}\right)\end{aligned} \tag{4.23}$$

One can easily check that the pole in ε is regularized taking $Z_n(g,\varepsilon)=1-K_n g/\varepsilon$ with $K_n=n(n-1)S_d/4$. This yields the following anomalous dimension:

$$\gamma_n(g)=K_n g+\mathcal{O}(g^2) \tag{4.24}$$

and in particular at the WF fixed point:

$$\gamma_n(g_*)=n(n-1)/6\varepsilon+\mathcal{O}(\varepsilon^2) \tag{4.25}$$

4.6 Conformal invariance in the LRI fixed point

We start by providing a test of conformal invariance, following [2]. This consists in calculating a two-point correlation function and ensuring that it respects the CFT constraints given in the first section of this work (we'll calculate one example, the second can be found in [2, 19]). After having motivated conformal invariance using this test, we prove it to all orders in perturbation theory by making use of the Ward identities.

4.6.1 A test of conformal invariance

Two-point correlation functions of identical operators aren't enough to differentiate between conformal and scale invariance, since both predict the same form. However, only the former predicts the vanishing of two-point functions of *distinct* primaries. As for a pair of descendants in CFT, their two-point function will be non-zero provided that they can be written as $\left(\partial^2\right)^n\mathcal{O}$ for the same primary $\mathcal{O}$. Since this implies scaling dimensions differing by an even integer, a good test of conformal invariance might be to study the two-point function of two operators whose dimensions do not differ by an even integer.

At the LRI fixed point, ϕ^3 can't be a descendent of ϕ by dimensional argument. We'll therefore prove that the correlation function $\langle\phi\phi^3\rangle$ cancels perturbatively in ε. The UV dimension of ϕ^3 being $3\Delta_\phi$ (ignoring a potential anomalous dimension), we can write

$$F(x,g,\mu):=\langle\phi(x)\left[\phi^3\right](0)\rangle=\frac{f(s,g)}{|x|^{\Delta_\phi+3\Delta_\phi}} \tag{4.26}$$

where $\left[\phi^3\right]$ is the renormalized operator derived in section 4.5 and $s=\mu|x|$ is a dimensionless variable. This function should be a a solution of the Callan-Symanzik equation for correlation functions (see section 3.5). This leads to the following equation for f:

$$\left[s\partial_s+\beta(g)\partial_g+\gamma_3(g)\right]f(s,g)=0 \tag{4.27}$$

It is worth noting that in the IR, the solution to the Callan-Symanzik equation above becomes

$$f(s,g)\approx Cs^{-\gamma_3(g_*)} \tag{4.28}$$

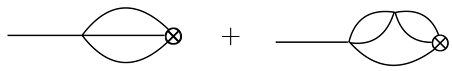
since $\beta(g)\to 0$ and $\gamma_3(g)\to\gamma_3(g_*)$ (this is a similar story to the end of section 3.5). However, the general solution to this Callan-Symanzik equation (whose derivation is found in [20]) is

$$f(s,g)=\hat{f}(\overline{g}(s,g))\exp\left(-\int_1^s d\ln s'\gamma_3\left(\overline{g}(s',g)\right)\right) \tag{4.29}$$

where, in keeping with [2], $\overline{g}$ is the running coupling defined by:

$$\begin{cases} s\partial_s\overline{g}=-\beta(\overline{g})\\ \overline{g}_{|s=1}=g\end{cases} \tag{4.30}$$

Since $f(s=1,g)=\hat{f}(g)$, the prefactor C defined in the IR by equation (4.28) is given by $C=\hat{f}(g_*)$. We want to show that $C=\mathcal{O}(\varepsilon^2)$ here. To do so, we start by computing the two-point function before renormalization $F_0(x)$. To second order in the coupling, keeping the combinatorial factors implicit:

$$F_0(x) := \langle \phi(x)\phi^3(0)\rangle = $$  $$\tag{4.31}$$

To carry out these computations, the following integrals are useful (see appendix D.1 for values of the w coefficients):

$$\int \frac{d^d y}{|x-y|^A |y|^B} = \frac{w_A w_B}{w_{A+B-d}} \frac{1}{|x|^{A+B-d}} \tag{4.32}$$

Indeed, one can then use the fact that $G(x-y) = |x-y|^{-2\Delta_\phi}$ and Wick's theorem to arrive at:

$$F_0(x) = \frac{R_1 g_0}{|x|^{d-2\varepsilon}} + \frac{R_2 g_0^2}{|x|^{d-3\varepsilon}} + \mathcal{O}(g_0^3) \tag{4.33}$$

where $R_2 = \mathcal{O}(1)$ and $R_1 = \mathcal{O}(\varepsilon)$ (see appendix D.2 for full computation). We now include renormalization of the coupling and of the composite field ϕ^3:

$$\begin{cases} g_0 = Zg\mu^\varepsilon = \left[g + Kg^2\varepsilon^{-1} + \mathcal{O}(g^3)\right]\mu^\varepsilon \\ \phi^3 = Z_3\phi^3 = \left[1 - K_3 g\varepsilon^{-1} + \mathcal{O}(g^2)\right]\left[\phi^3\right] \end{cases} \tag{4.34}$$

This allows us to compute the renormalized correlation function up to second order in g:

$$F(x) = \frac{gR_1 s^\varepsilon + g^2\left[R_1(K+K_3)\varepsilon^{-1}s^\varepsilon + R_2 s^{2\varepsilon}\right]}{|x|^{d-\varepsilon}} \tag{4.35}$$

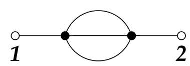
Setting $s = 1$ and using the explicit values of the various constants, one can extract the value of the prefactor at the fixed point:

$$C = \hat{f}(g_*) = -R_1\varepsilon^{-1}(-\varepsilon + Kg_*^2) = -R_1\varepsilon^{-1}\beta(g_*) = 0 \tag{4.36}$$

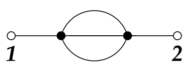
This is what we were meant to show. Next, we generalize this result to all orders in perturbation theory. To do so, we start by using the equations of motion of the theory:

$$\mathcal{N}_\sigma \mathcal{L}_\sigma \phi + \frac{g_0}{3!}\phi^3 = 0 \tag{4.37}$$

This entails that $\langle \phi^3(x)\phi(0)\rangle \propto \mathcal{L}_\sigma \langle \phi(x)\phi(0)\rangle$, hence one can use the usual two-point function of the theory to extract information on the correlation function in question. Since the field does not acquire an anomalous dimension in one-loop renormalization, its two-point function can be written as $\langle\phi(x)\phi(0)\rangle = \rho(x)|x|^{-2\Delta_\phi}$ where $\rho(x) = \hat{\rho}(\overline{g}(s,g))$. To second order in perturbation theory, the two-point function is given by (see figure 5 for relevant connected diagrams):

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta_\phi}} + \frac{1}{6}$$  $$\tag{4.38}$$

Since this diagram is slightly simpler than the computations involving composite operators, let's calculate it explicitly:



$$\begin{aligned} &= g^2 \int d^d y_1 d^d y_2 G(y_1 - x_1) G(y_1 - y_2)^3 G(x_2 - y_2) \\ &= g^2 \int d^d y_1 d^d y_2 \frac{1}{|y_1 - x_1|^{2\Delta_\phi}} \frac{1}{|y_1 - y_2|^{6\Delta_\phi}} \frac{1}{|x_2 - y_2|^{2\Delta_\phi}} \\ &= g^2 \int d^d y_1 d^d y_2 \frac{1}{|y_1 - (x_1 - y_2)|^{2\Delta_\phi}} \frac{1}{|y_1|^{6\Delta_\phi}} \frac{1}{|x_2 - y_2|^{2\Delta_\phi}} \\ &= g^2 \frac{w_{2\Delta_\phi} w_{6\Delta_\phi}}{w_{8\Delta_\phi - d}} \int d^d y_2 \frac{1}{|x_1 - y_2|^{8\Delta_\phi - d}} \frac{1}{|x_2 - y_2|^{2\Delta_\phi}} \\ &= g^2 \frac{w_{2\Delta_\phi}^2 w_{6\Delta_\phi}}{w_{10\Delta_\phi - 2d}} \frac{1}{|x_1 - x_2|^{10\Delta_\phi - 2d}} \\ &= g^2 \frac{\pi^d \Gamma\left(-\frac{d}{4}\right)}{\Gamma\left(\frac{3d}{4}\right)} \frac{1}{|x_1 - x_2|^{2\Delta_\phi}} \end{aligned} \tag{4.39}$$

In the last line, we've used the fact that we're close to the crossover $\sigma = d/2$ and this expression behaves smoothly near that boundary. Incidentally, this result supports conformal invariance, since it shows that the two-point function respects the conformal constraints in the first non-trivial correction. It's also worth noting that this would not be the case for arbitrary values of σ. Hence, we can write

$$\hat{\rho}(g) = 1 + Qg^2 \tag{4.40}$$

with $Q = (\pi^d/6)\Gamma(-d/4)/\Gamma(3d/4)$. Therefore, at short distances/high energy, $\rho(x) \to 1$ since $g \to 0$ (the interaction is relevant so negligible in the UV). At long distaces/low energy, by definition $g = g_* = \mathcal{O}(\varepsilon)$ hence $\rho(x) = 1 + \mathcal{O}(\varepsilon^2)$. Next, we want to describe the behavior of ρ as we approach the IR. To do so, let's consider a scale μ_c at which the coupling $g \sim g_*/2$, and let's linearize the β-function close to g_*, considering $\overline{g} = g_* + \delta g$. This yields

$$s\frac{d\delta g}{ds} = (\varepsilon - 2Kg_*)\delta g = -\varepsilon\delta g \tag{4.41}$$

and thus $\delta g \sim \varepsilon s^{-\varepsilon}$ (the factor ε comes from choosing $g \sim g_*/2$). It follows that one can write

$$\overline{g} = g_* - \mathcal{O}\left(\varepsilon s^{-\beta'(g_*)}\right) \tag{4.42}$$

where $\beta'(g_*) = \varepsilon + \mathcal{O}(\varepsilon^2)$, and hence close to the IR (equation (4.40) allows us to write the following):

$$\rho(x) = \rho(g_*) + \mathcal{O}\left(\varepsilon^2 s^{-\beta'(g_*)}\right) \tag{4.43}$$

We are now in a position to study the fractional Laplacian applied to the two-point function of ϕ. Defining the position-space kernel of the fractional Laplacian $T(x-y) \propto |x-y|^{-d-\sigma}$ (to be compared with equation (4.8)):

$$\begin{aligned}\mathcal{L}_\sigma\langle\phi(x)\phi(0)\rangle &\propto \int d^d y T(x-y)\rho(y)|y|^{-2\Delta_\phi} \\ &= \int d^d y T(x-y)\rho(g_*)|y|^{-2\Delta_\phi} + \int d^d y T(x-y)(\rho(y)-\rho(g_*))|y|^{-2\Delta_\phi}\end{aligned} \tag{4.44}$$

For large x, the first integral cancels exactly (by definition of the kernel, it is proportional to $\delta(x)$), while in the second the main contribution will come from large y. Hence the asymptotics in equation (4.43) apply:

$$\mathcal{L}_\sigma\langle\phi(x)\phi(0)\rangle \sim \int d^d y T(x-y)\varepsilon^2 s^{-\beta'(g_*)}|y|^{-2\Delta_\phi} \propto \frac{1}{|x|^\alpha} \tag{4.45}$$

where $\alpha = \sigma + 2\Delta_\phi + \beta'(g_*)$ (this integral is proportional to the type of integral calculated in appendix D.1). Next, recall equation (4.25), which implies $\gamma_3(g_*) = \varepsilon + \mathcal{O}(\varepsilon^2)$ and hence:

$$\begin{aligned}\alpha - (\Delta_\phi + \Delta_{\phi^3}) &= \sigma + 2\Delta_\phi + \beta'(g_*) - \Delta_\phi - 3\Delta_\phi - \gamma_3(g_*) \\ &= 2\sigma - d = \varepsilon + \mathcal{O}(\varepsilon^2) > 0\end{aligned} \tag{4.46}$$

Therefore $\langle\phi(x)\phi^3(0)\rangle$ obeys two distinct scaling laws at the LRI fixed point. This necessarily entails $C = 0$ exactly, which concludes the proof that this correlator vanishes. This is in line with conformal invariance in the LRI fixed point, although this correlator might seem special, in that it involves two operators which are closely linked to each other via the equations of motion. One could also show that a more generic correlator, such as $\langle\phi^2(x)\phi^4(0)\rangle$, vanishes. This is done in [2] and follows the exact same method as far as perturbative cancelation of the correlator is concerned.

4.6.2 Perturbative proof of conformal invariance in the LRI fixed point

So far we have been verifying that the key consequences of conformal invariance hold in the LRI fixed point. In this section, we go over the key steps to actually proving conformal invariance perturbatively in the ε expansion.

Caffarelli-Silvestre trick. To begin with, we show that the Gaussian theory is equivalent to a higher dimensional theory which is conformal – this is referred to as the Caffarelli-Silvestre trick in [2] (the original details are found in [21]). Consider a scalar field $\Phi(x, y)$ where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^p$ (both spaces are endowed with the usual Euclidean metric) with $p = 2 - \sigma$ (we suppose the results we derive below hold when σ is not an integer). Consider the following action:

$$S_{\text{CS}} = \int d^d x d^d y \left[(\partial_x \Phi)^2 + (\partial_y \Phi)^2\right] \tag{4.47}$$

defining $\phi(x) = \Phi(x, 0)$, such that the usual space we work in is a hyperplane embedded in a $d+p$-dimensional flat space. Let's also assume we can restrict ourselves to $\Phi(x, y) = \Phi(x, |y|)$. In spherical coordinates, this action can hence be written as

$$S_{\text{CS}} = S_p \int d^d x dz z^{1-\sigma} \left[(\partial_x \Phi)^2 + (\partial_z \Phi)^2\right] \tag{4.48}$$

The Euler-Lagrange equations yield

$$\partial_x^2 \Phi + \frac{1-\sigma}{z} \partial_z \Phi + \partial_z^2 \Phi = 0 \tag{4.49}$$

Going to momentum space relative to x gives

$$-p^2 \hat{\Phi} + \frac{1-\sigma}{z} \partial_z \hat{\Phi} + \partial_z^2 \hat{\Phi} = 0 \tag{4.50}$$

For small z, the solution to the above equation can be expanded as

$$\hat{\Phi}(p, z) = \left(1 + K\left(|p|z\right)^{\frac{\sigma}{2}} + \mathcal{O}(z^2)\right) \hat{\phi}(p) \tag{4.51}$$

Next, we perform integration by parts in the Caffarelli-Silvestre action and move to momentum space:

$$\begin{aligned} S_{\text{CS}} = -S_p \int d^d x dz z^{1-\sigma} \Phi \partial_z \Phi &\propto \int d^d x d^d p d^d q dz z^{1-\sigma} \hat{\Phi}(p, z) \partial_z \hat{\Phi}(p, z) e^{i(p+q)x} \\ &\propto \int d^d p d^d q dz z^{1-\sigma} \hat{\Phi}(p, z) \partial_z \hat{\Phi}(-p, z) \end{aligned} \tag{4.52}$$

For small z, we can plug expression (4.51) into the above action:

$$S_{\text{CS}} \propto \int d^d p \hat{\phi}(p) |p|^\sigma \hat{\phi}(-p) \tag{4.53}$$

It turns out that this corresponds exactly to the GFF part of the continuum LRI action (see equation (B.1) in the appendix).

Defect QFT. Now that we've embedded the free theory into a larger space, we consider the following interacting theory:

$$S = \frac{1}{2} \int d^{\bar{d}} X \left(\partial_M \Phi\right)^2 + \frac{g_0}{4!} \int_{y=0} d^d x \Phi^4 \tag{4.54}$$

where we keep the CS field Φ from the previous paragraph, and define the embedded space described $X = (x, y)$ in $\bar{d} = d + p$ dimensions. The Φ^4 interaction lives exclusively on the $y = 0$ hyperplane, and is what is referred to as a *defect*. For the remainder of this subsection, capital Latin letters run over the $\bar{d}$-dimensional space, lowercase Latin letters over the p-dimensional "perpendicular" subspace, and lowercase Greek letters over the d-dimensional "parallel" subspace.

Conformal Ward identities. Contrary to the LRI, the defect QFT has a stress tensor. The canonical stress tensor of the theory can easily be derived using equation (2.40):

$$T_{MN} = \partial_M \Phi \partial_N \Phi - \frac{1}{2} \delta_{MN} \left(\partial_K \Phi\right)^2 - \delta_{MN}^{||} \delta^{(p)}(y) \frac{g_0}{4!} \Phi^4 \tag{4.55}$$

where $\delta_{MN}^{||}$ is the Kronecker delta in the parallel subspace. We also need to compute the trace and divergence of this tensor for what follows:

$$\begin{cases} \partial^M T_{MN} = -E_N + \delta^{(p)}(y) D_N \\ T^M{}_M = -\Delta_\phi E - \varepsilon \frac{g_0}{4!} \delta^{(p)}(y)\Phi^4 + \left(\frac{1}{2} - \frac{\bar{d}}{4}\right) \partial_K^2 \Phi^2 \end{cases} \tag{4.56}$$

where we define the following new operators:

$$\begin{cases} E = \Phi\left[-\partial_K^2 \Phi + \delta^{(p)}(y) \frac{g_0}{3!}\Phi^3\right] \\ E_N = \partial_N \Phi \left[-\partial_K^2 \Phi + \delta^{(p)}(y)\frac{g_0}{3!}\Phi^3\right] \\ D_N = \frac{g_0}{3!}\Phi^3 \partial_n \Phi \text{ if } N = n \end{cases} \tag{4.57}$$

It turns out that the $\Phi^4 = \phi^4$ factors appearing at $y = 0$ need to be renormalized, and thus acquire an anomalous dimension. One can show that $\phi^4 = Z_4[\phi^4]$ with $Z_4 = -\beta(g)\mu^\varepsilon/(\varepsilon g_0)$ in the ε-expansion. Taking into account this wavefunction renormalization of ϕ^4 will prove crucial to the final argument. Using expressions (2.51) from section 2.7:

$$\begin{cases} j_D^M = T^{MN} X_N \\ j_K^{ML} = T^{MN}\left(2X_N X^L - \delta_N^L X^2\right) \end{cases} \tag{4.58}$$

We calculate the divergence of the currents for scale and special conformal transformations respectively:

$$\begin{cases} \partial_M j_D^M = -X^M E_M + T^M{}_M \\ \partial_M j_K^{ML} = \left(2X^N X^L - \delta^{NL} X^2\right)\left(E_N + \delta^{(p)}(y) D_N\right) + 2X^L T^M{}_M \end{cases} \tag{4.59}$$

Inserting these divergences into n-point functions leads to the two following broken Ward identities:

$$\sum_{i=1}^n \left[X_i \cdot \partial_{X_i} + \Delta_\phi\right] G(X_1, \ldots, X_n) = \beta(g)\frac{\mu^\varepsilon}{4!}\int d^d x G(X_1 \ldots X_n; \left[\phi^4\right](x)) \tag{4.60}$$

$$\sum_{i=1}^n \left[\left(2X_i^\mu X_i^\lambda - \delta^{\mu\lambda} X_i^2\right)\frac{\partial}{\partial X_i^\mu} + 2\Delta_\phi X_i^\lambda\right] G(X_1, \ldots, X_n) = 2\beta(g)\int d^d x x^\lambda G(X_1 \ldots X_n; \left[\phi^4\right](x)) \tag{4.61}$$

To prove conformal invariance, we should make use of the above identities after taking $y \to 0$. However, let's assume we can swap the limits and first take $g \to g_*$ (this is non-trivial, and further discussed in [2]). Since the β-function vanishes, this implies that the Ward identities hold for the full space X coordinates. Finally, taking $y \to 0$ allows us to translate these identities into Ward identities for the LRI fixed point. This proves that the LRI fixed point is conformally invariant to all orders in perturbation theory.

5 Conclusion

The LRI is a rich model, whose study has motivated a brief overview of conformal field theory, quantum field theory and the renormalization group, which are central to modern high energy physics. It has two interesting crossover points: the LRI and SRI fixed points. In this work, we've extensively reviewed the proof of conformal invariance in the LRI fixed point in [2], which uses a GFF perturbed by a ϕ^4 interaction. The other crossover point, $\sigma = \sigma^*$, can also be realized as a fixed point of a GFF + ϕ^4 flow, and proof of conformal invariance very similarly amounts to showing that the global conformal Ward identities hold. Incidentally, these arguments could apply to more generic models which are obtained by perturbing a GFF and using an ε-expansion.

However, there is a complementary picture presented in [22, 23] which we have not presented here for the sake of brevity. A first approach to the SRI crossover might be to perturb the free SRI action with the non-local fractional Laplacian term extensively studied in this work. However, conceptual difficulties arise from the fact that the perturbation is non-local and one moves from a phase with a stress tensor to a phase without one. A modified scheme has been suggested to take care of this issue, which considers the crossover from the SRI coupled to a "remnant" LRI GFF field to the intermediate non-Gaussian regime.

Finally, while beyond the scope of this work, it would have been interesting to delve deeper in our study of CFT. Topics to look at in more detail next include CFT in $d = 2$ dimensions, the AdS/CFT correspondence, defects and the conformal bootstrap. In fact, the latter is very relevant to the LRI, with recent work focusing on bootstrapping the model [24].

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A Wick's theorem

Since Wick's theorem is of prime importance in QFT, we supply a justification of the statement here, following [13]. One way to derive Wick's theorem is via a generating functional defined by

$$Z_0[j] = \int \mathcal{D}\phi e^{-\frac{1}{2}\int d^dxd^dy\left(\phi(x)D(x,y)\phi(y) - \int d^dxj(x)\phi(x)\right)} \tag{A.1}$$

where j is a source, or forcing term (comparable to current in the QED action, or to the magnetic field in the Ising model). We also use the notation $Z_0 = Z_0[j=0]$ for the partition function in this subsection. Correlation functions of the free-field theory can then be obtained by taking functional derivatives of $Z_0[j]$ at $j=0$:

$$G_0^{(n)}(x_1,\ldots,x_n) := \left[\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\frac{Z_0[j]}{Z_0}\right]_{j=0} \tag{A.2}$$

We can compute $Z_0[j]$ to derive all n-point functions of the free theory. This can be done using the following change of variables:

$$\phi(x) \to \phi'(x) = \phi(x) - \int d^dx' D^{-1}(x,x')j(x') \tag{A.3}$$

where we've used the functional inverse of D, with the δ-function behaving as the identity:

$$\int d^dx' D(x,x')D^{-1}(x',x'') = \delta(x-x'') \tag{A.4}$$

This entails

$$\begin{aligned}\int d^dxd^dy\phi(x)D(x,y)\phi(y) = &\int d^dxd^dy\phi'(x)D(x,y)\phi'(y) + \int d^dxd^dyd^dx'D^{-1}(x,x')j(x')D(x,y)\phi'(y)\\ &+\int d^dxd^dyd^dx'\phi'(x)D(x,y)D^{-1}(y,x')j(x')\\ &+\int d^dxd^dyd^dx'd^dx''D^{-1}(x,x')j(x')D(x,y)D^{-1}(y,x'')j(x'')\\ =&\int d^dxd^dy\phi'(x)D(x,y)\phi'(y) + 2\int d^dx\phi'(x)j(x)\\ &+\int d^dxd^dx'j(x)D^{-1}(x,x')j(x')\end{aligned} \tag{A.5}$$

&

$$\int d^dx\phi(x)j(x) = \int d^dx\phi'(x)j(x) + \int d^dxj(x)D^{-1}(x,x')j(x') \tag{A.6}$$

Note that $D^{-1}(x,x')$ is essentially a propagator, so we assume it commutes with the other terms (this is the case for the LRI, for Klein-Gordon theory for example). This yields the following transformation of the energy functional (with a source):

$$E[\phi] \to E[\phi'] - \frac{1}{2}\int d^dxd^dx'j(x)D^{-1}(x,x')j(x') \tag{A.7}$$

Hence the generating functional of the theory becomes:

$$Z_0[j] = \int \mathcal{D}\phi' e^{-E_0[\phi']}\exp\left(\frac{1}{2}\int d^dxj(x)D^{-1}(x,x')j(x')\right) = Z_0\exp\left(\frac{1}{2}\int d^dxj(x)D^{-1}(x,x')j(x')\right) \tag{A.8}$$

Taking functional derivatives and dividing by the partition function Z_0 leads us to the following expression for n-point functions:

$$G_0^{(n)}(x_1,\ldots,x_n) = \left[\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\exp\left(\frac{1}{2}\int d^dxd^dx'j(x)D^{-1}(x,x')j(x')\right)\right]_{j=0} \tag{A.9}$$

By expanding the exponential, one will notice that the only non-zero contribution comes from the $n/2$-th term in the series, since we are taking n derivatives and the integrand is quadratic in j. Hence:

$$G_0^{(n)}(x_1,\ldots,x_n) = \frac{1}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\left[\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\left(\int d^dx d^dx' j(x) D^{-1}(x,x') j(x')\right)^{\frac{n}{2}}\right]_{j=0} \tag{A.10}$$

The case $n = 2$ leads to $G^{(2)}(x_1,x_2) = D^{-1}(x_1,x_2) = G(x_1 - x_2)$ i.e. the Green's function of D (the propagator). Using the above equation, one can convince oneself that after evaluating the functional derivatives, a sum of products of propagators remains. This yields Wick's theorem, which is crucial to the main computations in this work (and QFT):

$$G^{(n)}(x_1,\ldots,x_n) = \sum_{\text{all pairings}} G(x_{i_1} - x_{i_2})\ldots G(x_{i_{n-1}} - x_{i_n}) \tag{A.11}$$

B The Fractional Laplacian

B.1 S_0 and the fractional Laplacian

In this additional section, we justify why the free Gaussian theory of the continuum LRI can be written using a fractional Laplacian $\mathcal{L}_\sigma = \left(-\partial^2\right)^{\sigma/2}$. Going to Fourier space and back:

$$\begin{aligned}
S' &:= \int d^dx \phi(x)\left(-\partial^2\right)^{\frac{\sigma}{2}}\phi(x) \\
&= \frac{1}{(2\pi)^{2d}}\int d^dx d^dk d^dl \hat{\phi}(k)\hat{\phi}(l) k^\sigma e^{i(k+l)x} \\
&= \frac{1}{(2\pi)^{d}}\int d^dk d^dl \hat{\phi}(k)\hat{\phi}(-k) k^\sigma \\
&= \frac{1}{(2\pi)^{d}}\int d^dk d^dx d^dy \phi(x)\phi(y) e^{ik(y-x)} k^\sigma
\end{aligned} \tag{B.1}$$

To give an estimate of the integral over k, one can use hyperspherical coordinates. Since we only care about finding what S' is proportional to, we need not worry about the angular part of the integral over the $(d-1)$-sphere. This yields:

$$S' = |S_{d-1}|\int dk d\theta \sin\theta k^{d+\sigma} e^{ik|x-y|\cos\theta} \tag{B.2}$$

Doing the θ integral, integrating by parts $d+\sigma$ times and introducing a UV cutoff (here a maximum wave vector norm) gives:

$$\int d^dk e^{ik(y-x)} k^\sigma \propto \frac{1}{|x-y|^{d+\sigma}} \tag{B.3}$$

which, plugged into the last line of (B.1), yields the desired result.

B.2 Conformal transformation law for the fractional Laplacian

In the main text, we argued for conformal invariance of the free theory using its propagator. Usually, when one is asked to prove some symmetry holds in a given theory, one studies the effect it has on the action. Therefore, let's show that the GFF action S_0 is conformally invariant (this is also shown in [25]). The quasi-primary field ϕ in this model transforms in the following way under conformal transformations:

$$\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_\phi}{d}}\phi(x) \tag{B.4}$$

where $\Delta_\phi = (d-\sigma)/2$ as mentioned previously. One can then show that the fractional Laplacian transforms in the following manner:

$$\mathcal{L}'_\sigma\phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_\phi}{d}-1}\mathcal{L}_\sigma\phi(x) \tag{B.5}$$

To show this, one needs to check how the action changes under translations, rotations, scale transformations and SCTs. As an example, let's check the case of SCTs, since this is the most intricate. Finite SCTs are given by equation (2.12). Let's ignore the normalization of the fractional Laplacian for this calculation.

$$\mathcal{L}'_\sigma \phi'(x') = \int d^d y' \frac{\phi'(y')}{\left|\frac{x - x^2 b}{1-2(b\cdot x)x + b^2x^2} - y'\right|^{d+\sigma}} \tag{B.6}$$

From equation (2.12), one can easily show that for a SCT:

$$\Lambda(x) = \left(1 - 2(b\cdot x)x + b^2 x^2\right)^2 \tag{B.7}$$

Hence:

$$\begin{aligned}\mathcal{L}'_\sigma \phi'(x') &= \Lambda(x)^{\frac{d+\sigma}{2}} \int d^d y' \frac{\phi'(y')}{\left|x - \left(x^2 b + \sqrt{\Lambda(x)}y'\right)\right|^{d+\sigma}} \\ &= \Lambda(x)^{\frac{d+\sigma}{2}} \Lambda(x)^{-\frac{d}{2}} \int d^d y \frac{\phi'\left(\Lambda(x)^{-\frac{1}{2}}(y - x^2 b)\right)}{|x-y|^{d+\sigma}}\end{aligned} \tag{B.8}$$

Using ϕ's scaling dimension Δ_ϕ:

$$\begin{aligned}\mathcal{L}'_\sigma \phi'(x') &= \Lambda(x)^{\frac{\Delta+\sigma}{2}} \int d^d y \frac{\phi\,(y)}{|x-y|^{d+\sigma}} \\ &= \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_\phi}{d}-1} \mathcal{L}_\sigma \phi(x)\end{aligned} \tag{B.9}$$

This implies that

$$\phi'(x')\mathcal{L}'_\sigma \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-1} \mathcal{L}_\sigma \phi(x) \tag{B.10}$$

So that, having recognized a Jacobian on the RHS, integrating against x' yields

$$\int d^d x' \phi'(x')\mathcal{L}'_\sigma \phi'(x') = \int d^d x \phi(x) \mathcal{L}_\sigma \phi(x) \tag{B.11}$$

One can proceed in a similar way, and this is in fact easier, for the remaining conformal transformations. Therefore, the Gaussian theory is conformal by direct calculation.

C $\phi^n \times \phi^4$ OPE

A key ingredient in the derivation of the β-function of GFF + ϕ^4 is the $\phi^4 \times \phi^4$ OPE in the free theory. One can also use the $\phi^n \times \phi^4$ OPE to derive the wavefunction renormalization and anomalous dimension of ϕ^n. In this section, we study the latter OPE, since it contains the former. Suppose we have the following OPE for $x \to 0$

$$\phi^n(x) \times \phi^4(0) = \sum_k \frac{\lambda_{n4k}}{|x|^{n\Delta_\phi + 4\Delta_\phi - \Delta_k}} \mathcal{O}_k(0) \tag{C.1}$$

where the $\mathcal{O}_k(x)$ account for all of the operators of the theory, typically powers of ϕ and its derivatives. Note that $\Delta_{\phi^n} = n\Delta_\phi$ in the free theory. Next, we insert this OPE into the following three point function:

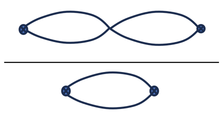
$$\langle \phi^n(x)\phi^4(0)\phi^n(y)\rangle = \sum_k \frac{\lambda_{n4k}}{|x|^{n\Delta_\phi + 4\Delta_\phi - \Delta_k}} \langle \mathcal{O}_k(0)\phi^n(y)\rangle \tag{C.2}$$

Ignoring the differential operators arising from the presence of descendants, the only contribution comes from $\mathcal{O}_k = \phi^n$ since the two-point function vanishes otherwise.

$$\langle \phi^n(x)\phi^4(0)\phi^n(y)\rangle = \frac{\lambda_{n4n}}{|x|^{4\Delta_\phi}} \langle \phi^n(0)\phi^n(y)\rangle \tag{C.3}$$

Thus λ_{n4n} can be calculated using a ratio of correlation functions. Normal-ordering removes diagrams with $G(0)$ factors (tadpoles), there are only two classes of relevant diagrams:

$$\frac{\lambda_{n4n}}{|x|^{4\Delta_\phi}} = $$



(C.4)

The bold points in the above Feynman diagrams are ϕ^n insertions, and the lines are to be understood as representing multiple lines, since after applying Wick's theorem we are left with propagators. To count the number of diagrams in the numerator, we count the number of ways to associate one of the central ϕ operators to the operators on the extremities, and then the number of ways to pair the remaining operators on the left vertex with the remaining ones on the right vertex. This is given by:

$$\binom{4}{2} \times \frac{n!}{(n-2)!} \times \binom{2}{2} \frac{n!}{(n-2)!} \times (n-2)! = 6n(n-1)n! \tag{C.5}$$

The combinatorial factor for the diagram in the denominator is simply given by the number of permutations of n fields i.e. $n!$. Hence, $\lambda_{n4n} = 6n(n-1)$, and we arrive at the statement we were meant to prove when $x \to 0$:

$$\phi^n(x) \times \phi^4(0) \supset \frac{6n(n-1)}{|x|^{d-\varepsilon}} \phi^n(0) \tag{C.6}$$

where we recall that in the ε-expansion $4\Delta_\phi = 2(d-\sigma) = d - \varepsilon$.

D Correlation function calculations

D.1 Derivation of useful integrals

When calculating correlation functions to motivate conformal invariance in the LRI fixed point, we used the following identity:

$$I(x) := \int \frac{d^d y}{|x-y|^A |y|^B} = \frac{w_A w_B}{w_{A+B-d}} \frac{1}{|x|^{A+B-d}} \tag{D.1}$$

where $w_A = (4\pi)^{d/2} 2^{-A} \Gamma\left(\frac{d-A}{2}\right) / \Gamma\left(\frac{A}{2}\right)$. Since it is a key ingredient in previous computations and it is easily derived using the Gaussian theory's propagator, we provide a derivation in this section. The main idea is to calculate it in momentum space, and then perform an inverse Fourier transform back to position space. The Fourier transform of I obeys

$$\hat{I}(p) := \int \frac{d^d x d^d y e^{-ipx}}{|x-y|^A |y|^B} = \int \frac{d^d x d^d y e^{-ip(x+y)}}{|x|^A |y|^B} = (2\pi)^{2d} G_A(p) G_B(p) \tag{D.2}$$

where we've applied a translation by $+y$ to x, and $G_A(p) = K_A |p|^{A-d}$ is the Gaussian propagator for $\sigma = A$. Transforming back to position space yields

$$I(x) = (2\pi)^{2d} K_A K_B \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{|p|^{2d-A-B}} = (2\pi)^{2d} K_A K_B K_{2d-A-B} \frac{1}{|x|^{A+B-d}} \tag{D.3}$$

Using the values of the K coefficients derived in the main text leads to the desired result.

D.2 Composite operator diagrams

In the main text, we skipped the detailed calculation of diagram (4.31). Let's go through it here for the sake of completeness.

First diagram. The first diagram is given by:

$$F_1 := -4! \times \frac{g_0}{4!} \int d^d y \frac{1}{|x-y|^{-2\Delta_\phi}} \frac{1}{|y|^{6\Delta_\phi}} \tag{D.4}$$

where we've taken into account the combinatorial factor (there are 4! ways to associate the legs of the ϕ^4 interaction with the external operators). By using the class of useful integrals derived just previously, this yields in the ε-expansion:

$$F_1 = -g_0 \frac{w_{\frac{d-\varepsilon}{2}} w_{3\frac{d-\varepsilon}{2}}}{w_{d-2\varepsilon}} \frac{1}{|x|^{d-2\varepsilon}} \tag{D.5}$$

Before taking $\varepsilon \to 0$, notice that $w_{d-2\varepsilon}$ has a pole in ε. This stems from the expansion of the Γ-function near 0. Indeed, for $s \to 0$:

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1) = \frac{1}{s} + o\left(\frac{1}{s}\right) \tag{D.6}$$

Therefore when $\varepsilon \to 0$:

$$w_{d-2\varepsilon} = (4\pi)^{\frac{d}{2}} 2^{-d} \frac{\Gamma(\varepsilon)}{\Gamma\left(\frac{d}{2}\right)} \sim \frac{\pi^{\frac{d}{2}}}{\varepsilon\Gamma\left(\frac{d}{2}\right)} \tag{D.7}$$

The other factors have a limit at $\varepsilon \to 0$, hence:

$$\begin{aligned} F_1 &\sim -g_0\varepsilon\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}\right) w_{\frac{d}{2}} w_{\frac{3d}{2}} \frac{1}{|x|^{d-2\varepsilon}} = -g_0\varepsilon\pi^{\frac{d}{2}} \frac{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{3d}{4}\right)} \frac{1}{|x|^{d-2\varepsilon}} \\ &= \frac{R_1 g_0}{|x|^{d-2\varepsilon}} \end{aligned} \tag{D.8}$$

where $R_1 = \mathcal{O}(\varepsilon)$ as announced in the main text. It's worth noticing that this contribution to the correlator is conformal when $\varepsilon \to 0$. Indeed, it goes like $|x|^{-d} = |x|^{-\Delta_\phi - \Delta^{(0)}_{\phi^3}}$ where $\Delta_\phi = d/4$ and $\Delta^{(0)}_{\phi^3} = 3d/4$ (no anomalous dimension yet), and since this is effectively the correlator of two operators with distinct scaling dimensions, it vanishes since $R_1 = \mathcal{O}(\varepsilon)$, in accordance with the behavior of two-point functions in CFT.

Second diagram The second diagram is the first non-trivial contribution in the ε expansion. To calculate it, we go to second order in the bare coupling g_0. First of all, let's calculate the combinatorial factor. At each vertex, we choose which fields ϕ are linked together and then permutate the links. There are $\binom{4}{2}^2$ ways of picking the four fields involved in the first loop on the left, and 2! ways to permutate the links. For the second loop, there are $\binom{2}{2} \times \binom{3}{2}$ ways of picking the fields (only two remaining from the central ϕ^4 vertex, and three to choose from the composite operator) and again 2! permutations. Lastly, there are 2 fields remaining from the leftmost internal vertex, which can be connected in 2! ways to the outer vertices. This yields the following factor in front of the integral over internal vertex positions:

$$\frac{g_0^2}{4!^2} \times \binom{4}{2}^2 \times 2! \times \binom{2}{2} \times \binom{3}{2} \times 2! \times 2! = \frac{3}{2} g_0^2 \tag{D.9}$$

Therefore, the full expression of the diagram is given by

$$F_2 := \frac{3g_0^2}{2} \int \frac{d^d y_1 d^d y_2}{|x-y_1|^{2\Delta_\phi} |y_1 - y_2|^{4\Delta_\phi} |y_1|^{2\Delta_\phi} |y_2|^{4\Delta_\phi}} \tag{D.10}$$

Again, this can easily be evaluated using the integrals derived in this section of the appendix. Evaluating the y_2 integral, followed by the y_1 integral yields in the ε-expansion:

$$F_2 = \frac{3g_0^2}{2} \frac{w_{d-\varepsilon}^2 w_{\frac{d-\varepsilon}{2}} w_{\frac{3d-5\varepsilon}{2}}}{w_{d-2\varepsilon} w_{d-3\varepsilon}} \frac{1}{|x|^{d-3\varepsilon}} \tag{D.11}$$

When $\varepsilon \to 0$, the $w_{d+k\varepsilon}$ have simple poles in ε. However, these cancel since they have equal contributions from the numerator and the denominator. The other factors have well-defined limits, and performing the computation yields:

$$F_2 \sim 9\pi^d g_0^2 \frac{\Gamma\left(-\frac{d}{4}\right)}{\Gamma\left(\frac{3d}{4}\right)} \frac{1}{|x|^{d-3\varepsilon}} = \frac{R_2 g_0^2}{|x|^{d-3\varepsilon}} \tag{D.12}$$

with $R_2 = \mathcal{O}(1)$.

E Useful Feynman diagrams

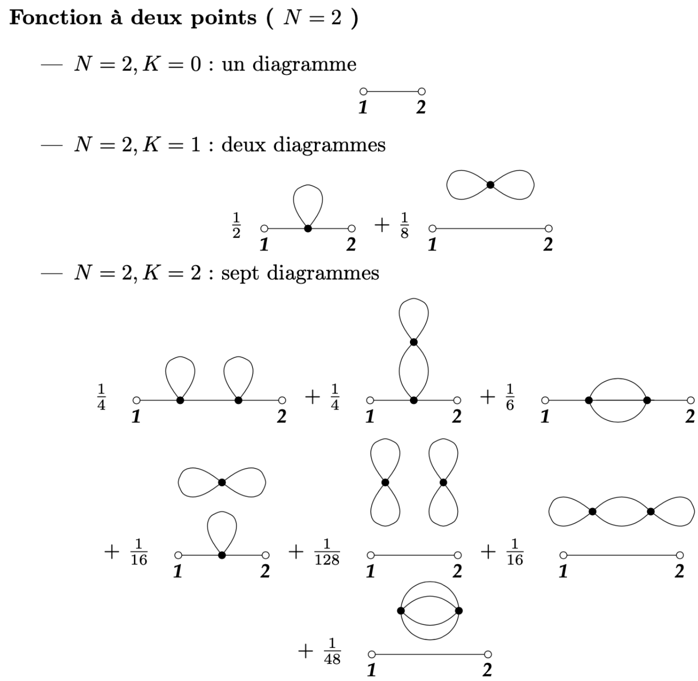


Figure 4: Feynman diagrams involved in the computation of the two-point function up to second order in perturbation theory for a ϕ^4 interaction. The disconnected diagrams cancel after dividing by the partition function. Taken from [10].

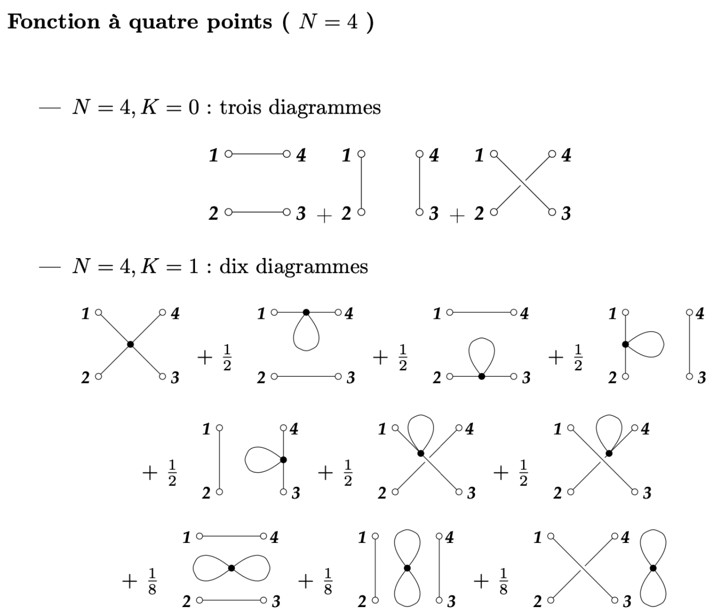


Figure 5: Feynman diagrams involved in the computation of the four-point function up to first order in perturbation theory for a ϕ^4 interaction. Taken from [10].